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On the alternating use of "unanimity" and "surjectivity" in the Gibbard–Satterthwaite Theorem☆

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Abstract

Surjectivity and unanimity can be equivalently used to state the Gibbard–Satterthwaite Theorem. On the other hand, over restricted domains, replacing surjectivity with unanimity makes a stronger statement.

Keywords: Strategy-proofness; Manipulation; Unanimity; Surjectivity; Gibbard–Satterthwaite Theorem

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1. Overview

We know since Gibbard (1973) and Satterthwaite (1975) that when a society confronts at least three alternatives, there exists no surjective, non-dictatorial and strategy-proof social choice function defined over the universal domain of preference profiles. This result—to which we refer as the "GS Theorem"—paved the way to a rich literature on exploring strategy-proof social choice rules. Some of the research in this area[1] quotes the GS Theorem as "the nonexistence of a unanimous, non-dictatorial and strategy-proof social

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[1] For example Aswal et al. (2003), Ozyurt and Sanver (2006).

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choice function defined over the universal domain of preference profiles." We refer to this latter statement as the "modified version of the GS Theorem".

The GS Theorem and its modified version are logically equivalent, as

(i) any unanimous social choice function defined over the universal domain of preference profiles is surjective — hence the GS Theorem implies its modified version.
(ii) any surjective and strategy-proof social choice function defined over the universal domain of preference profiles is unanimous — hence the modified version of the GS Theorem implies its original version.

However, the universal domain assumption is critical for statement (i) to hold while the truth of statement (ii) does not depend on the domain. Thus, when the GS Theorem is stated over restricted domains, its modified version is stronger than its original version.

After giving the basic concepts in Section 2, we make our point formally in Section 3 and conclude in Section 4.

2. Basic concepts

Taking any integer $n \geq 2$, consider a society $\mathbf{N}=\{1,\ldots, n\}$ and a finite set of alternatives $\mathbf{A}$ with $\#\mathbf{A} \geq 3$. We write Π for the set of complete, transitive and antisymmetric binary relations over $\mathbf{A}$. Every $\rho \in \Pi$ stands for some *preference* over $\mathbf{A}$, which we write as ρ_i when it belongs to a particular $i \in \mathbf{N}$. A *preference profile* is an n-tuple $\underline{\rho} \in \Pi^{\mathbf{N}}$ of individual preferences. Letting $D \subseteq \Pi$ stand for an arbitrary non-empty subdomain of Π, we define a *social choice function* (SCF) as a mapping f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$. A SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is *surjective* iff given any $x \in \mathbf{A}$, there exists $\underline{\rho} \in D^{\mathbf{N}}$ such that $f(\underline{\rho})=x$. A SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is *unanimous* iff for all $x \in \mathbf{A}$ and for all $\underline{\rho} \in D^{\mathbf{N}}$ with $x \rho_i y \, \forall y \in \mathbf{A}$, $\forall i \in \mathbf{N}$, we have $f(\underline{\rho})=x$. A domain D is *regular* iff given any $x \in \mathbf{A}$, there exists $\rho \in D$ such that $x \rho y \, \forall y \in \mathbf{A}$.[2]

A SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is *manipulable* by $i \in \mathbf{N}$ at $\underline{\rho} \in D^{\mathbf{N}}$ iff there exists $\underline{\rho}' \in D^{\mathbf{N}}$ with $\rho_j'=\rho_j$ for all $j \in \mathbf{N} \setminus \{i\}$ such that $f(\underline{\rho}') \neq f(\underline{\rho})$ and $f(\underline{\rho}') \rho_i f(\underline{\rho})$. A SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is *strategy-proof* iff f is manipulable at no $\underline{\rho} \in D^{\mathbf{N}}$, by no $i \in \mathbf{N}$. A SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is *dictatorial* iff there exists $d \in \mathbf{N}$ such that $f(\underline{\rho})=\rho_d$ at each $\underline{\rho} \in D^{\mathbf{N}}$.

We say that a domain D is *γ-dictatorial* iff D admits no surjective, non-dictatorial and strategy-proof SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$.[3] Similarly, a domain D is *δ-dictatorial* iff D admits no unanimous, non-dictatorial and strategy-proof SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$.

A SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is *Maskin monotonic* iff given any $x \in \mathbf{A}$ and any $\underline{\rho}$, $\underline{\rho}' \in \Pi^{\mathbf{N}}$ such that $x \rho_i y \Rightarrow x \rho_i' y \, \forall y \in \mathbf{A}$, $\forall i \in \mathbf{N}$, we have $f(\underline{\rho})=x \Rightarrow f(\underline{\rho}')=x$.

3. Results

Lemma 3.1. *Take any $D \subseteq \Pi$. Every strategy-proof SCF f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is Maskin monotonic.*[4]

Proof. Let f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ be strategy-proof. Remark that for all $x \in \mathbf{A}$ and all $\underline{\rho}$, $\underline{\rho}' \in \Pi^{\mathbf{N}}$ such that $x \rho_i y \Rightarrow x \rho_i' y \, \forall y \in \mathbf{A}$ for some $i \in \mathbf{N}$ and $\rho_j'=\rho_j$ for all $j \in \mathbf{N} \setminus \{i\}$, we have $f(\underline{\rho})=x \Rightarrow f(\underline{\rho}')=x$, as otherwise

[2] For a SCF defined over a regular domain, unanimity implies surjectivity.

[3] When D is not regular, any surjective f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$ is non-dictatorial. Thus, for a non-regular D, the definition of γ-dictatoriality becomes equivalent to the non-existence of surjective and strategy-proof SCFs over D.

[4] Lemma 3.1 states that one side of the Muller and Satterthwaite (1977) equivalence between strategy-proofness and the monotonicity condition of Maskin (1999) holds independent of the domain over which SCFs are defined.

f would be manipulable by $i \in \mathbf{N}$ at $\underline{\rho}$ or $\underline{\rho}'$. Applying the argument to all $j \in \mathbf{N}$ establishes the Maskin-monotonicity of f. □

Theorem 3.1. *[D is δ-dictatorial ⇒ D is γ-dictatorial] holds for any* $D \subseteq \Pi$.

Proof. Take any $D \subseteq \Pi$ which is not γ-dictatorial. So there exists a surjective, non-dictatorial and strategy-proof f: $D^{\mathbf{N}} \to \mathbf{A}$. We complete the proof by showing that f is unanimous, hence D is not δ-dictatorial. Take any $x \in \mathbf{A}$. In case there exists no $\rho \in \Pi$ with $x\rho y \; \forall y \in \mathbf{A}$, what unanimity requires is trivially satisfied. Now consider the case where there exists $\rho \in \Pi$ with $x\rho y \; \forall y \in \mathbf{A}$. As f is surjective, there exists $\underline{\rho} \in D^{\mathbf{N}}$ such that $f(\underline{\rho})=x$. Take some $\underline{\rho}' \in D^{\mathbf{N}}$ with $x\rho_i' y \; \forall y \in \mathbf{A}$, $\forall i \in \mathbf{N}$. As f is Maskin monotonic by Lemma 3.1, we have $f(\underline{\rho}')=x$ and in fact $f\underline{\rho}''=x$ for all $f\underline{\rho}'' \in D^{\mathbf{N}}$ with $x\rho_i'' y \; \forall y \in \mathbf{A}$, $\forall i \in \mathbf{N}$, showing the unanimity of f. □

Theorem 3.2. *[D is δ-dictatorial ⇔ D is γ-dictatorial] holds for any* $D \subseteq \Pi$ *which is regular.*

Proof. Take any $D \subseteq \Pi$ which is regular. We have [D is δ-dictatorial ⇒ D is γ-dictatorial] by Theorem 3.1. To see that [D is γ-dictatorial ⇒ D is δ-dictatorial], suppose D is not -dictatorial. So there exists a unanimous, non-dictatorial and strategy-proof f: $D^{\mathbf{N}} \to \mathbf{A}$. As every unanimous SCF defined over a regular domain is surjective, f also shows that D fails to be γ-dictatorial. □

So δ-dictatoriality of a domain is generally stronger than its γ-dictatoriality, while the two concepts coincide over regular domains.[5] Nevertheless, regularity is not necessary for this coincidence. We illustrate this through Examples 1 and 2 below. Finally, we give Example 3, which is an instance where δ-dictatoriality is stronger than γ-dictatoriality.

Example 1. A non-regular domain D which is neither δ-dictatorial nor γ-dictatorial.

Take $\mathbf{N}=\{1, 2\}$ and $\mathbf{A}=\{a, b, c\}$. Consider the domain $D=\{\rho, \rho'\}$ where $a\rho b\rho c$ and $c\rho' b\rho' a$. We know from Theorem 5.2 of Aswal et al. (2003) that D is not δ-dictatorial.[6] One can see that D is not γ-dictatorial either, by checking that the surjective and non-dictatorial SCF f: $D^{\mathbf{N}} \to \mathbf{A}$ defined as $f(\rho, \rho)=a$, $f(\rho, \rho')=b$, $f(\rho', \rho)=b$, $f(\rho', \rho')=c$ is strategy-proof.

Example 2. A non-regular domain D which is both δ-dictatorial and γ-dictatorial.

Take $\mathbf{A}=\{a, b, c, d\}$. Consider the domain $D=\{\rho \in \Pi: x\rho d$ for all $x \in \mathbf{A}\}$. Remark that given any strategy-proof f: $D^{\mathbf{N}} \to \mathbf{A}$, if $f(\underline{\rho})=d$ for some $\underline{\rho} \in D^{\mathbf{N}}$, then $f(\underline{\rho})=d$ for all $\underline{\rho} \in D^{\mathbf{N}}$. Thus D exhibits no surjective and strategy-proof SCF, showing its γ-dictatoriality. To see that D is δ-dictatorial as well, suppose f: $D^{\mathbf{N}} \to \mathbf{A}$ is a unanimous, non-dictatorial and strategy-proof SCF. As we have just remarked, $f(\underline{\rho}) \in \{a, b, c\}$ for all $\underline{\rho} \in D^{\mathbf{N}}$. Now take the subset $\mathbf{B}=\{a, b, c\}$ of $\mathbf{A}$. Let $\Pi_{\mathbf{B}}$ be the set of complete, transitive and antisymmetric binary relations over $\mathbf{B}$. For each $\rho \in \Pi$, let $\rho_{\mathbf{B}} \in \Pi_{\mathbf{B}}$ be the restriction of ρ over $\mathbf{B}$, i.e., $x\rho_{\mathbf{B}} y \Leftrightarrow x\rho y$ for all $x, y \in \mathbf{B}$. We write $D_{\mathbf{B}}=\{\rho_{\mathbf{B}} \in \Pi_{\mathbf{B}}: \rho \in D\}$. Consider the function $f_{\mathbf{B}}$: $[D_{\mathbf{B}}]^{\mathbf{N}} \to \mathbf{B}$ defined for each $\rho_{\mathbf{B}} \in D_{\mathbf{B}}$ as $f_{\mathbf{B}}(\rho_{\mathbf{B}})=f(\rho)$. As f: $D^{\mathbf{N}} \to \mathbf{A}$ is unanimous, non-dictatorial and strategy-proof, $f_{\mathbf{B}}$: $[D_{\mathbf{B}}]^{\mathbf{N}} \to \mathbf{B}$ is unanimous, non-dictatorial and strategy-proof as well. Thus, $D_{\mathbf{B}}$ is not δ-dictatorial which contradicts the Gibbard–Satterthwaite Theorem, as $D_{\mathbf{B}}=\Pi_{\mathbf{B}}$.

[5] For example Π is regular and although the original statement of the Gibbard–Satterthwaite Theorem is about the γ-dictatoriality of Π, one can equivalently state it as the δ-dictatoriality of Π.

[6] This theorem says that when $\#\mathbf{A}=3$, a domain D is δ-dictatorial iff $D=\Pi$.

Example 3. A (non-regular) domain D which is γ-dictatorial but not δ-dictatorial.[7]

Take $\mathbf{N}=\{1, 2\}$ and $\mathbf{A}=\{a, b, c\}$. Consider the domain $D=\{\rho, \rho'\}$ where $a\rho b\rho c$ and $b\rho' c\rho' a$. We know from Theorem 5.2 of Aswal et al. (2003) that D is not δ-dictatorial.[8] To see that D is γ-dictatorial, suppose there exists a surjective, non-dictatorial and strategy-proof f: $D^{\mathbf{N}} \rightarrow \mathbf{A}$. As f is surjective, there exists $\underline{\rho} \in D^{\mathbf{N}}$ such that $f(\underline{\rho})=c$. First suppose $f(\rho, \rho)=c$. By strategy-proofness, we have $f(\rho', \rho)=f(\rho, \rho')=c$, contradicting the surjectivity of f. Next suppose $f(\rho', \rho)=c$. By strategy-proofness, we have $f(\rho, \rho) \in \{a, c\}$ and $f(\rho', \rho')=c$. By surjectivity, we have $f(\rho, \rho)=a$ and $f(\rho, \rho')=b$. However, f is manipulable by 1 at (ρ', ρ'). Supposing $f(\rho, \rho')=c$ leads to a similar contradiction. Finally suppose $f(\rho', \rho')=c$. By strategy-proofness, we have $f(\rho, \rho') \in \{a, c\}$ and $f(\rho', \rho) \in \{a, c\}$. By surjectivity, we have $f(\rho, \rho)=b$. Now strategy-proofness implies $f(\rho, \rho')=f(\rho', \rho)=c$, violating surjectivity. Thus there exists no $\underline{\rho} \in D^{\mathbf{N}}$ such that $f(\underline{\rho})=c$, contradicting that f is surjective.

4. Conclusion

When social choice functions operating over the universal domain are considered, one can harmlessly replace surjectivity with unanimity in the original version of the GS Theorem. On the other hand, the analysis of strategy-proof social choice functions defined over restricted domains requires some caution in the alternating use of surjectivity and unanimity.

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[7] By Theorem 3.2, any domain exemplifying a case where δ-dictatoriality is stronger than γ-dictatoriality is non-regular, hence, exhibits an example of a domain over which surjectivity and strategy-proofness are incompatible (see Footnote 4).

[8] See Footnote 7.