Population Monotonicity in Fair Division of Multiple Indivisible Goods[1]

Emre Doğan[§]

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Abstract

We consider the fair division of a set of indivisible items where each agent can get more than one good and monetary transfers are allowed. For the problems with three or more goods, population monotonicity is incompatible with efficiency except for very specific Cartesian product preference domains. For the 2-goods case, Shapley solution and the constrained egalitarian solution is PM on the subadditive preference domain. We also define hybrid solutions that are PM on the full domain. Among them, the hybrid Shapley solution is PM.

Key Words: *Population monotonicity, fair division, indivisible goods*

1 Introduction

We consider the fair division problem where individuals have equal claims on a set of indivisible items, and each agent can get more than one good. We also allow for balanced monetary transfers among agents.

Many authors have studied population monotonicity for the fair division problem (Moulin (1990, 1992), Beviá (1996a,b), Tadenuma and Thomson (1993)). It is mostly interpreted as a solidarity principle. Among the two well-known versions, upon the arrival of an additional agent, the stronger notion (PM) asks no one to be better-off, while the weaker one (wPM) requires everyone to be affected in the same direction; either everyone loose or everyone gains.

The essence of the solidarity idea, indeed, lies in the weaker version. In case there is no production, no monetary transfers among the agents, and agents have monotone preferences,

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[§] Department of Economics, Rice University, Houston, TX 77251-1892, USA
email: emre.dogan@rice.edu

when an extra claimant appears in the allocation process of a fixed supply of consumption goods, unambiguously, he creates a burden on the existing agents. There, both PM and wPM asks no existing agents to be better-off. However, if utility is transferable, an extra claimant who receives more utility from some bundle that is not very desirable for the existing agents would possibly be beneficial to the society. Therefore, in our setting wPM is more suitable as a solidarity principle.

In a very general setting where there is no restriction on the individual preferences, Beviá (1996a) showed that PM is incompatible with efficiency for problems with more than 4 goods. Moreover, PM is incompatible with one of the most important fairness criterion; envy-freeness (Alkan 1994, Moulin 1990). Then, the question is: "Why should we care about population monotonic solutions?"

Consider fair division of a single object among three agents, and the equal division solution which is obviously wPM. Let the valuations of individuals be 2, 4 and 9, respectively. Each agent receives 3 units of utility. However, if agent 3 does not appear, agent 2 receives the object and a monetary transfer of -2. A redistribution of his allocation among agents 2 and 3 yields 7 units of utilities, which leaves room for both to be better-off. Here, agents 2, and 3 would manipulate the outcome by leaving 3 outside the allocation process.

Doğan (2013) defines the cooperative manipulation concept above, and the corresponding stability concept absence-proofness[2]. He also showed that population monotonic solutions are stable in that sense.

Here, we put forward the stability aspect of population monotonicty besides the solidarity aspect, and try to make an extensive analysis of population monotonicity in this setting.

Aside from the negative result, Beviá (1996a) also showed that when the domain of preference profiles satisfies "substitutability", the induced transferable utility (TU) game is concave and hence the Shapley solution (Shapley 1962) is PM (Sprumont (1990)). However, substitutability is not defined on the Cartesian product of individual preference domains. Assuming free disposability (monotone preferences), we try to stretch Beviá's both the positive and the negative results for different number of goods under different Cartesian product domains where individual preferences are submodular, subadditive, and superadditive.

In Section 2, we give the general setting. In Section 3, we define some well-known individual preference domains and basic fairness and stability properties: symmetry, continuity, equal split guarantee; population monotonicity and absence-proofness. In Section 4, we define two well-known symmetric and continuous solutions, the Shapley solution and the constrained egalitarian (Dutta and Ray (1989)), which are also PM in concave games. In Section 5, we analyze problems with three or more goods. Here, PM and efficiency are incompatible on the superadditive domain. Also, neither submodular nor subadditive preferences induce concave TU games. Moreover, the Shapley solution is not PM on any of these domains.

In Section 6, we first show that when each agent has subadditive preferences the Shapley solution and the egalitarian solution are PM. On the full domain of monotone preferences we can write the efficient surplus as the summation of surplus derived from two problems where

[2] See Section 3 for a formal definition.

each problem induces concave TU games: a 2-goods problem with subadditive preferences and a single good problem. We define hybrid solutions as the summation of two solutions to those problems. However, dynamics of a change in population is not trivial in our construction. A hybrid solution is PM if both solutions are PM and solution to the single good problem satisfies additive scale monotonicity[3]. The hybrid Shapley solution is PM on the full domain while the hybrid egalitarian solution is not. Finally, we show that the equal split guarantee is not compatible with PM for problems with two or more goods.

2 The setting

A finite set of commonly owned indivisible goods denoted by Ω is distributed to a set of individuals denoted by $N \in \mathcal{N}$ where $\mathcal{N}$ is the set of all finite potential societies and $|N| = n$. We consider a general model where agents can get multiple goods. Monetary transfers are available, and agents' preferences are quasilinear in money. Given (N, Ω), each agent $i \in N$ receives $u_i(A)$ units of utility from the bundle $A \subseteq \Omega$, while also receiving $m \in \mathbb{R}$ units of money yields $u_i(A, m) = u_i(A) + m$. We also assume free disposability (monotone preferences), i.e., $u_i(A) \leq u_i(B)$ for all $A \subseteq B$. By convention $u_i(\emptyset) = 0$. A list of preferences $\{u_i\}_{i \in N}$ is denoted by u, and u^S denotes the restricted list to $S \subseteq N$. A *fair division problem* is a triple (N, Ω, u), ε denotes the set of all problems (with monotone preferences), and ε_m denotes all problems with m goods.

Given a problem with monetary transfers, an allocation consists of two components: Assignment of the objects to the agents and balanced monetary transfers among the agents. An *assignment* is a mapping $\sigma: N \to 2^{\Omega}$ such that $\sigma_i \cap \sigma_j = \emptyset$ for all $i, j \in N$, and $\bigcup_{i \in N} \sigma_i = \Omega$, while some agents may receive no good with $\sigma_i = \emptyset$. A vector of *balanced monetary transfers* is $m \in \mathbb{R}^n$ s.t. $\sum_{i \in N} m_i = 0$.

An assignment σ is *efficient* if $\sum_{i \in N} u_i(\sigma_i) \geq \sum_{i \in N} u_i(\sigma'_i)$ for all assignments $\sigma': N \to 2^{\Omega}$. By quasilinearity, an allocation (σ, m) is efficient if and only if σ is efficient. Here, we are only interested in individually rational (IR) allocations, i.e., $u_i(\sigma_i) + m_i \geq 0$ for all $i \in N$. A *solution* φ is a mapping such that in a given domain of problems, it assigns a set of allocations to each problem. φ is efficient (or IR) if it always yields efficient (or IR) allocations, and is *single valued* if for any (N, Ω, u), for any $(\sigma, m), (\sigma', m') \in \varphi(N, \Omega, u)$ we have $u_i(\sigma_i) + m_i = u_i(\sigma'_i) + m'_i$ for all $i \in N$. Then, $\varphi^i(N, \Omega, u)$ denotes the final utility of agent i at solution φ. Throughout this paper, for simplicity, we will write down some properties and results for single valued solutions only. However, this simplification does not alter any result stated here.

Each problem (N, Ω, u) induces a *TU cooperative game* $v: 2^N \to \mathbb{R}_+$ with $v(S) = \sum_{i \in S} u_i(\sigma_i)$ where σ is an efficient assignment at the problem (S, Ω, u^S). We also write $v(S, A) = \sum_{i \in S} u_i(\sigma_i)$ if σ is an efficient assignment at the problem (S, A, u^S) with $A \subseteq \Omega$. By convention $v(\emptyset) = 0$. A game v is *concave* if $v(S \cup \{i, j\}) - v(S \cup \{j\}) \leq v(S \cup \{i\}) - v(S)$ for all $S \subseteq N$, and $i, j \in N \setminus S$.

[3] See Section 6 for a definition.

3 Preferences and basic properties

Throughout this paper we will consider the following basic types of preferences:

Definition 1: A utility function $u_i: 2^{\Omega} \to \mathbb{R}_+$

(i) is *submodular* if for all $A, B \subseteq \Omega$, $u_i(A \cup B) + u_i(A \cap B) \leq u_i(A) + u_i(B)$.

(ii) has *decreasing marginal returns* if for all $A \subseteq B \subseteq \Omega$, $\alpha \in A$,

$$u_i(B) - u_i(B \setminus \{\alpha\}) \leq u_i(A) - u_i(A \setminus \{\alpha\}).$$

(iii) is *subadditive* if for all $A, B \subseteq \Omega$, $u_i(A \cup B) \leq u_i(A) + u_i(B)$.

(iv) is *additively separable* if for all $A \subseteq \Omega$, $u_i(A) = \sum_{\alpha \in A} u_i(\alpha)$.

(v) is *superadditive* if for all $A, B \subseteq \Omega$, $u_i(A) + u_i(B) \leq u_i(A \cup B)$.

Properties (ii) and (iii) are equivalent, and reflect the concavity of u_i (see e.g. Gul and Stacchetti (1999)). Subadditive preference domain contains the submodular preferences. Both subadditive and superadditive preference domain contains the additively separable preferences. In case of a single object, all properties trivially hold. Also, for $|\Omega| = 2$, subadditivity and submodularity coincides.

Definition 2: A solution φ is

(i) *population monotonic* (*PM*) on $\varepsilon' \subseteq \varepsilon$ if for all $(N', \Omega, u), (N, \Omega, u^N) \in \varepsilon'$ with $N \subseteq N'$, and for all $i \in N$, we have $u_i(\sigma_i) + m_i \geq u_i(\sigma_i') + m_i'$ for all $(\sigma', m') \in \varphi(N', \Omega, u)$ and $(\sigma, m) \in \varphi(N, \Omega, u^N)$.

(ii) *weakly population monotonic* (*wPM*) on $\varepsilon' \subseteq \varepsilon$ if for all $(N', \Omega, u), (N, \Omega, u^N) \in \varepsilon'$ with $N \subseteq N'$, and for all $(\sigma', m') \in \varphi(N', \Omega, u)$ and $(\sigma, m) \in \varphi(N, \Omega, u^N)$ we have either $u_i(\sigma_i) + m_i \leq u_i(\sigma_i') + m_i'$ for all $i \in N$ or $u_i(\sigma_i) + m_i \geq u_i(\sigma_i') + m_i'$ for all $i \in N$.

Doğan (2013) defines the following cooperative manipulation argument and the corresponding stability concept absence-proofness.

Manipulation by absence. Given a problem (N, Ω, u) and a solution φ, a group of agents $T \subseteq N$ renounce their claims. The solution proposes a set of allocations at the reduced problem $\left(N \setminus T, \Omega, u^{N \setminus T}\right)$. Let $K \subseteq N \setminus T$ meet with agents in T after the allocation process in the reduced problem and reallocate the sum of goods and money they received at one of the allocations proposed by φ among $K \cup T$. By doing so, if their total utility is greater compared to the total utility at some allocation proposed by φ at the original problem, φ is said to be manipulable by $K \cup T$ via absence of T.

Absence-proofness. A solution φ is absence-proof if it is not manipulable at any problem by any group of agents.

Proposition 1: (*Doğan 2013*) If a solution φ is population monotonic, then it is also absence-proof.

Doğan (2013) also shows that wPM is not a sufficient condition for absence-proofness. In this work we put forward the stability aspect of population monotonicity besides the solidarity aspect, and therefore use PM as our main principle. Where we have positive results, along with the PM we also look for solutions with basic fairness criteria. The most basic one is that a solution does not discriminate agents by their names and cares for only their preferences. We also do not want any significant jumps in the final utilities when there is a miniscule change in the utility profile.

Symmetry. A *single valued* solution φ satisfies symmetry if for any two problems (N, Ω, u) and (N, Ω, u') s.t. $u_i = u'_j$, $u_j = u'_i$ for some $i, j \in N$, and $u_k = u'_k$ for all $k \in N \setminus \{i, j\}$ we have; $\varphi^i(N, \Omega, u) = \varphi^j(N, \Omega, u')$, $\varphi^j(N, \Omega, u) = \varphi^i(N, \Omega, u')$, and $\varphi^k(N, \Omega, u) = \varphi^k(N, \Omega, u')$ for all $k \in N \setminus \{i, j\}$.

Continuity. A *single valued* solution φ is continuous if for any fixed (N, Ω), φ^i is a continuous function of u.

Another important fairness idea is that no agent is worse off compared to receiving $(1/n)^{\text{th}}$ of the goods. For the indivisible goods, $(1/n)^{\text{th}}$ is not well defined. Beviá (1996c) discusses this issue and uses identical preference lower bound (IPLB) to apply the idea. In our case, for each agent this lower bound corresponds to $(1/n)^{\text{th}}$ of the efficient surplus generated at the hypothetical problem where everybody else has the same preferences. Here, we ask for a less demanding lower bound.

Equal Split Guarantee (ESG). A *single valued* solution φ satisfies ESG if for all $i \in N$, $\varphi^i(N, \Omega, u) \geq u_i(\Omega)/n$.

In the worst case scenario, an agent receives all the goods and then compensates others so that everybody gets the same final utility. In case $|\Omega| = 1$, where PM and ESG are compatible, ESG and IPLB coincides. When $|\Omega| \geq 2$, PM is not even compatible with the weaker property, ESG (see Section 7).

4 Concave TU games and solutions

In allocation problems where utility is transferable, an important source in the design of solutions is the set of algorithms that calculate the efficient surplus (or cost). However, excluding the special case where utilities are additively separable[4], finding the efficient surplus is not an easy task. Indeed, it is an NP hard problem for submodular preference domain (Feige (2009)). If symmetry and continuity is desired as a minimal fairness principle, a remedy is to borrow solutions from TU game literature such as the Shapley solution and the egalitarian solution (Dutta Ray (1989)).

Concave games are of special importance for two reasons. First, both solutions are population monotonic on concave games. Moreover, the egalitarian solution yields the unique vector that Lorenz dominates every other vector within the dual core of the game.

[4] Just assign each good to some agent who gets the maximum utility from that good.

The Shapley solution (Sh). Given a problem (N, Ω, u), $(\sigma, m) \in Sh(N, \Omega, u)$ if and only if for all $i \in N$, $u_i(\sigma_i) + m_i = Sh^i(v)$ where v is the induced TU game.[5]

Before we define the egalitarian solution, we need to define the dual core, which is closely related to population monotonicity.

Stand-alone Core (SAC). An allocation (σ, m) is in the SAC if $\sum_{i \in S} u_i(\sigma_i) + m_i \leq v(S)$ for all $S \subseteq N$. A single valued solution φ is in the SAC on $\varepsilon' \subseteq \varepsilon$ if for all $(N, \Omega, u) \in \varepsilon'$, and for all $S \subseteq N$, $\sum_{i \in S} \varphi^i(N, \Omega, u) \leq v(S)$.

Note that population monotonic solutions are always in the SAC as otherwise for some S, at least one agent from S is strictly worse-off when $N \setminus S$ leaves.

Dutta and Ray (1989) defined the Lorenz core and related egalitarian solution in the TU game context. They also defined an algorithm to calculate the solution for convex TU games. The dual algorithm for concave games, defined by Klijn et. al (2001), is as follows: Fix a concave TU game (N, v) and for any $S \subseteq N$, $e(S, v) = v(S)/|S|$, so that $e(S, v)$ is the average worth of S under v. Define $v_1 = v$.

Step 1: Define by S_1 the unique coalition such that (i) $e(S_1, v_1) \leq e(S, v_1)$ for all $S \subseteq N$; (ii) $|S_1| > |S|$ for all $S \neq S_1$ such that $e(S_1, v_1) = e(S, v_1)$; so that S_1 is the largest coalition with the lowest average worth (under concavity S_1 exists). Define

$$Eg^i(v) = e(S_1, v_1) \text{ for all } i \in S_1. \quad (1)$$

Step k: Suppose that $S_1 \dots S_{k-1}$ have been defined recursively and $\bigcup_{p=1}^{k-1} S_p \neq N$. Define a new game with the set of agents $N^k = N \setminus \bigcup_{p=1}^{k-1} S_p$. For all $S \subseteq N^k$, define $v_k(S) = v_{k-1}(S \cup S_{k-1}) - v_{k-1}(S_{k-1})$. This new game (N^k, v_k) is concave. Let S_k be the largest coalition with the lowest average worth and define

$$Eg^i(v) = e(S_k, v_k) \text{ for all } i \in S_k. \quad (2)$$

Now we define a sufficient condition for PM on the domain of preference profiles.

Substitutability. Given a problem (N, Ω, u), and $S \subseteq N$, $v(S,\cdot)$ satisfies substitutability if for all $A \subseteq B \subseteq \Omega$ and $\alpha \notin B$, $v(S, B \cup \{\alpha\}) - v(S, B) \leq v(S, A \cup \{\alpha\}) - v(S, A)$.

Proposition 2: (*Beviá 1996a*). If $\varepsilon^{subs} \subseteq \varepsilon$ is such that for each (N, Ω, u), $S \subseteq N$, $v(S,\cdot)$ satisfies substitutability, the induced game $v(\cdot)$ is concave for all $(N, \Omega, u) \in \varepsilon^{subs}$.

Corollary 1: The Shapley solution and the egalitarian solution are PM on ε^{subs}.

Both solutions are symmetric, and the Shapley value is continuous in v by definition. Egalitarian solution is also continuous in v (see e.g. Hougaard et al (2005)). As v is continuous in u, both solutions satisfy continuity.

[5] Given any TU game (N, v), $Sh^i(v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n-|S|-1)!}{n!} \big(v(S \cup i) - v(S)\big)$.

5 Problems with $|\Omega| \geq 3$

Here, we want to emphasize two main points. First one is that the incompatibility between the population monotonicity and efficiency is more serious than it was proven in Beviá (1996a). Secondly, we feel that the positive result (Proposition 2) needs some clarification in terms of the Cartesian product domain of profiles for which the result holds.

The incompatibility result in Beviá (1996a) is based on an example with 4 goods where individuals are allowed to have non-monotonic preferences (no free disposal). However, if we allow non-monotonic preferences, incompatibility prevails even in a 2-person, 2-good problem. Consider the following problem;

Example 1: $u_1(\alpha) = u_2(\beta) = 3, u_1(\beta) = u_2(\alpha) = 0, u_1(\alpha\beta) = u_2(\alpha\beta) = 2$

Note that $\varphi^1(N) + \varphi^2(N) = 6$ for any efficient solution. Hence, for at least one i we have $\varphi^i(N) \geq 3$. However, each agent gets 2 units of utility at the efficient solution when they are the only claimants. Thus, achieving PM is not possible here.

We strengthen this negative result in two dimensions. Incompatibility between PM and efficiency prevails for 3-goods problem even in economies with monotone preferences. Let ε^{sup} represent the set of problems where each agent has superadditive preferences.

Proposition 3: No efficient solution is population monotonic on $\varepsilon' \supseteq \varepsilon_m^{sup}$ for $m \geq 3$.

Proof: Let $|\Omega| = 3$, and consider the following preferences:

A	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
$u_1(A)$	0	0	0	2	0	0	2
$u_2(A)$	0	0	2	0	2	2	2
$u_3(A)$	0	0	0	0	2	0	2
$u_4(A)$	0	2	0	2	0	2	2

Consider the problem (S, Ω, u) with $S = \{1,2,3\}$. At the unique efficient assignment $\sigma_1 = \alpha\beta$, $\sigma_2 = \gamma$ and $v(S) = v(12) = 4$. Also, $v(13) = v(23) = 2$. Let φ be efficient, population monotonic and $(\sigma, m) \in \varphi(S, \Omega, u)$. By PM, (σ, m) is in the SAC. Then, $u_1(\sigma_1, m_1) + u_3(\sigma_3, m_3) \leq 2$, $u_2(\sigma_2, m_2) + u_3(\sigma_3, m_3) \leq 2$. Hence, we have $u_1(\sigma_1, m_1) = u_2(\sigma_2, m_2) = 2$, and $u_3(\sigma_3, m_3) = 0$. Then, $\varphi^1(\{1,3\}) = 2$ again by PM. Now, consider the problem (S', Ω, u) with $S' = \{1,3,4\}$. By a similar argument we have $\varphi^3(\{1,3\}) = 2$. Therefore, we have the desired contradiction. For $|\Omega| > 3$, use the same profile and add dummy goods such that it brings 0 extra utility to all bundles for all agents. □

Cartesian product of two special types of submodular preferences constitutes substitutable domains. First one is $u_i(A) = max_{\alpha \in A} u_i(\alpha)$ for all agents. Then, our problem is equivalent to the allocation problem where each agent can get at most one good. Moulin (1992) showed for this case that the utility profile is substitutable. Hence, the induced game is concave, and the Shapley and the egalitarian solutions are PM.

Additively separable preferences also constitute a substitutable domain. Given a problem (N, Ω, u) with u_i is additively separable for all $i \in N$, for a fixed $S \subseteq N$, marginal contribution of α to the efficient surplus $v(S,\cdot)$ is constant and equal to $max_{i\in S}\, u_i(\alpha)$. However, in this case, instead of the general approach of distributing many goods at once, we can just think of the problem as distributing $|\Omega|$ goods separately. The solution to the general problem can be defined as the sum of solutions to $|\Omega|$ separate problems. Obviously, if at each single good allocation problem the solution is PM, then their summation is also PM. Hence, for this domain of preferences there is a variety of efficient and PM solutions.[6]

A	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
$u_1(A)$	2	0	0	3	2	1	3
$u_2(A)$	0	2	0	2	0	2	2
$u_3(A)$	2	0	0	2	2	0	2

Table 1

We know that substitutability is a sufficient condition for the concavity of the induced game, and hence for the existence of a PM solution. It also requires that individual preferences are submodular. However, submodularity of the individual preferences is not a necessary condition for concavity. Although agent 1's preference is not submodular in the 3-person problem in Table 1, the induced game is concave[7].

What if all individuals have submodular preferences? Note that additively separable preferences constitute the border between the submodular (concave) and supermodular (convex) utility functions. At a first glance, if the least concave utility functions in submodular domain induce concave games, one may expect that when concavity becomes more severe, the induced game would be still concave. However, this intuition fails. Consider the 4-person, 3-goods problem in Table 2. The induced game is not concave as $v(N) - v(124) = 2 > v(123) - v(12) = 1$ while all agents have submodular utility functions. Also, concavity of the game is a necessary but not a sufficient condition for Shapley solution to be population monotonic[8]. The same example also illustrates that the Shapley solution is not PM on the domain of submodular preferences. Here, we have $Sh^3(\{1,2,3\}) \cong 4{,}67$, while $Sh^3(N) \cong 4{,}92$.

A	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
$u_1(A)$	8	4	8	12	14	8	14
$u_2(A)$	0	1	4	1	4	4	4
$u_3(A)$	8	0	0	8	8	0	8
$u_4(A)$	0	8	0	8	0	8	8

Table 2

[6] See Section 6.

[7] $v(1) = 3, v(2) = 2, v(3) = 2, v(12) = 4, v(13) = 3, v(23) = 4, v(123) = 4.$

[8] Consider the following game: $v(1) = 10, v(2) = 8, v(3) = 6, v(12) = 14, v(13) = 13, v(23) = 11, v(123) = 18$. Note that agent 1's contribution to $S = \{1,2\}$ is 6 while he contributes 7 to N. It is easy to check that the Shapley value of this game is population monotonic.

6 Problems with $|\Omega| \leq 2$

The problem of allocating a single indivisible object is studied by several authors. However, all population monotonic solutions to this problem (except the Shapley solution) are defined for the well-known airport (cost sharing) problem (see Thomson (2007) for a survey). This problem yields concave TU games and admits several population monotonic solutions other than the Shapley solution and the constrained egalitarian solution. It is easy to check that these two solutions also satisfy ESG when $|\Omega| = 1$. Our intention here is to work on the 2-goods case. However, to construct PM solutions for this case we need to define a property on solutions to the single good problem.

For the single good problem, preference of agent i is represented by a number u_i, and (N, u) represents a problem. Here, for simplicity we assume the ordered profile $u_1 \leq u_2 \ldots \leq u_n$. Efficiency dictates assigning the good to an agent with the highest u_i. Note that $v(N) = u_n$. Then, for any efficient, individually rational, and single valued solution we have $\sum_{i\in N} \varphi^i(N, u) = u_n$, and $\varphi^i \geq 0$ for all $i \in N$.

Given any ordered problem (N, u), by the convention that $u_0 = 0$, Shapley solution is calculated as follows:

$$Sh^i = \sum_{k=1}^{i} \frac{u_k - u_{k-1}}{|N| - (k-1)} \text{ for all } i \in N \tag{3}$$

Definition 3: A solution φ is *additively scale monotonic* (*ASM*) if for any $(N, u), (N, u') \in \varepsilon_1$ such that for some $t \in \mathbb{R}_+$, $u'_i = u_i + t$ for all $i \in N$, we have $\varphi^i(N, u') \geq \varphi^i(N, u)$ for all $i \in N$.

Lemma 1: The Shapley solution is ASM while the egalitarian solution is not.

Proof: Without loss of generality, take any ordered problem $(N, u) \in \varepsilon_1$, and $t \in \mathbb{R}_+$ so that (N, u') is defined as in the statement above. Note that the order is preserved at the profile u'. Then, for each $j > 1$, $u_j - u_{j-1} = u'_j - u'_{j-1}$, and $u_1 - u_0 + t = u'_1 - u'_0$. Therefore, $Sh^i(N, u') = Sh^i(N, u) + t/n$ for all $i \in N$, by (3).

To see that the egalitarian solution is not ASM, consider the following profiles for $n = 3$: $u = (0,0,2)$, and $u' = (1,1,3)$. Note that $Eg^3(N, u) = 2 > 1 = Eg^3(N, u')$. □

Additive scale monotonicity is a fairly week property. A trivial example of a PM solution that is also ASM is the equal distribution of the efficient surplus u_n among the agents with the highest valuation.

6.1 The case $|\Omega| = 2$

In this case, we have a more clear way of partitioning the monotone preferences in a useful way for the purpose of our analysis. Here, the monotone preferences are either superadditive, i.e., $u_i(\alpha) + u_i(\beta) \leq u_i(\alpha\beta)$, or subadditive, i.e., $max\{u_i(\alpha), u_i(\beta)\} \leq u_i(\alpha\beta) \leq u_i(\alpha) + u_i(\beta)$, or both (additively separable), i.e, $u_i(\alpha) + u_i(\beta) = u_i(\alpha\beta)$. Also, subadditivity and submodularity coincides. Moreover, they are not only necessary but also sufficient for substitutability. Now, let ε_2^{sub} be the set of all problems such that u_i is monotone and subadditive for all $i \in N$.

Proposition 6: For any $(N, \Omega, u) \in \varepsilon_2^{sub}$ the induced TU game v is concave.

Proof: Let $(N, \Omega, u) \in \varepsilon_2^{sub}$ and $S \subseteq N$. It suffices to show that $v(S,\cdot)$ satisfies substitutability. The only relevant case is $A = \emptyset$, and $B = \{\beta\}$. Hence, we need to show $v(S, \{\alpha\beta\}) - v(S, \{\beta\}) \leq v(S, \{\alpha\})$. Note that $v(S, \{\alpha\}) = max_{i \in S} u_i(\alpha)$, and $v(S, \{\alpha\beta\}) = u_i(\alpha) + u_j(\beta)$ for some distinct $i, j \in S$ or $v(S, \{\alpha\beta\}) = u_i(\alpha\beta)$ for some $i \in S$. Then, by definition, subadditivity implies $v(S, \{\alpha\beta\}) \leq max_{i \in S} u_i(\alpha) + \max_{i \in S} u_i(\beta)$. □

Corollary 2: The Shapley solution and the egalitarian solution are PM on ε_2^{sub}.

Unlike the general case $|\Omega| \geq 3$, here, even if some or all agents have superadditive preferences, we have population monotonic solutions. We now introduce some extra notation.

Fix a problem $(N, \Omega, u) \in \varepsilon_2$ and let $v(\cdot)$ be the induced TU game. For any $i \in N$, define $\bar{u}_i(\alpha\beta) \stackrel{\text{def}}{=} min\{u_i(\alpha\beta), u_i(\alpha) + u_i(\beta)\}$, while $\bar{u}_i(c) = u_i(c)$ for $c = \alpha, \beta$. Then, we have the perturbed problem $(N, \Omega, \bar{u})$ and the associated TU game $\bar{v}(\cdot)$. The only difference between the two problems is the utilities of agents who have (strictly) superadditive utilities at the original problem. For such an agent the only difference is $\bar{u}_i(\alpha\beta) = u_i(\alpha) + u_i(\beta) < u_i(\alpha\beta)$. Then, by construction, we have $(N, \Omega, \bar{u}) \in \varepsilon_2^{sub}$. Now, define $\tilde{u}_i \stackrel{\text{def}}{=} max(u_i(\alpha\beta) - \bar{v}(N), 0)$, and $\tilde{v}(N) \stackrel{\text{def}}{=} max_{i \in N} \tilde{u}_i$.

Lemma 2: For any $(N, \Omega, u) \in \varepsilon_2$, we have $v(N) = \bar{v}(N) + \tilde{v}(N)$.

Proof: Take any $(N, \Omega, u) \in \varepsilon_2$. Consider first the case $v(N) = u_j(\alpha) + u_k(\beta)$ for some distinct $j, k \in N$. Then, $u_i(\alpha\beta) \leq u_j(\alpha) + u_k(\beta)$ for all $i \in N$, and also $\bar{v}(N) = u_j(\alpha) + u_k(\beta)$. Hence, $\tilde{u}_i = 0$ for all $i \in N$ and $\tilde{v}(N) = 0$. Now, consider the case $v(N) = u_j(\alpha\beta)$ for some $j \in N$. If $u_j(\alpha\beta) = \bar{u}_j(\alpha\beta)$, again we have $\bar{v}(N) = v(N)$ and $\tilde{v}(N) = 0$. Now, let $u_j(\alpha\beta) > \bar{u}_j(\alpha\beta)$. Then, we have $\tilde{v}(N) = u_j(\alpha\beta) - \bar{v}(N)$. □

i	$u_i(\alpha) = \bar{u}_i(\alpha)$	$u_i(\beta) = \bar{u}_i(\beta)$	$u_i(\alpha\beta)$	$\bar{u}_i(\alpha\beta)$	$\tilde{u}_i$
1	2	2	10	4	2
2	0	4	9	4	1
3	4	4	7	7	0
4	2	2	6	4	0

Table 3

Consider the 4-person problem in Table 3 which clarifies our construction. Note that $v(N) = u_1(\alpha\beta) = 10$, $\bar{v}(N) = u_2(\alpha) + u_3(\beta) = 8$. An efficient, individually rational, and single valued solution to the problem (N, Ω, u) is just a nonnegative distribution of $v(N) = 10$ units of surplus to the 4 agents. We aim to write a solution as the summation of two solutions to two different problems. As the first component of our hybrid solution to the problem (N, Ω, u), we pick a solution $\bar{\varphi}(N, \Omega, \bar{u})$. Then, $\bar{v}(N) = 8$ units of surplus is distributed by $\bar{\varphi}$. By Lemma

2, the remaining to distribute is $\tilde{v}(N)$. Note that by definition, $\tilde{v}(N) = \tilde{u}_j$ for some $j \in N$ and in that case $\tilde{u}_j \geq \tilde{u}_i$ for all $i \in N$. Also, $\tilde{u}_i \geq 0$ for all $i \in N$. Then, we can think of distributing $\tilde{v}(N)$ as allocation of a single good where agents' valuations are $\tilde{u}_i$. Let us call this problem $(N, \tilde{u})$. Then, the remaining $\tilde{v}(N) = 2$ units of surplus is distributed by some $\tilde{\varphi}(N, \tilde{u})$.

Definition 4: Given any two solutions $\bar{\varphi}$ on ε_2^{sub}, and $\tilde{\varphi}$ on ε_1, φ is a hybrid (of $\bar{\varphi}$ and $\tilde{\varphi}$) solution on ε_2 if for all $(N, \Omega, u) \in \varepsilon_2$ we have, $(\sigma, m) \in \varphi(N, \Omega, u)$ if $u_i(\sigma_i, m_i) = \bar{\varphi}^i(N, \Omega, \bar{u}) + \tilde{\varphi}^i(N, \tilde{u})$ for all $i \in N$.

Note that a hybrid solution is single valued and well-defined by Lemma 2.

Proposition 7: Let φ be a hybrid solution such that for all $(N, \Omega, u) \in \varepsilon_2$, $i \in N$, $\varphi^i = \bar{\varphi}^i + \tilde{\varphi}^i$ where $\bar{\varphi}$ is a solution to $(N, \Omega, \bar{u})$, and $\tilde{\varphi}$ is a solution to $(N, \tilde{u})$.

(i) φ is efficient, symmetric and continuous if both $\bar{\varphi}$ and $\tilde{\varphi}$ are efficient, symmetric and continuous.

(ii) φ is population monotonic if both $\bar{\varphi}$ and $\tilde{\varphi}$ are population monotonic, and $\tilde{\varphi}$ is ASM.

Proof: We skip the trivial argument for efficiency and symmetry. Let $\bar{\varphi}$ and $\tilde{\varphi}$ be continuous in $\bar{u}$ and $\tilde{u}$, respectively. By construction $\bar{u}$ is continuous in u. Also, we know that $\bar{v}(N) = max\{max_{i \neq j}\bar{u}_i(\alpha) + \bar{u}_j(\beta), max_i \bar{u}_i(\alpha\beta)\}$ is a continuous function of $\bar{u}$, and hence, it is continuous in u. Then, by definition $\tilde{u} = \{max(u_i(\alpha\beta) - \bar{v}(N), 0)\}_{i \in N}$ is continuous in u. Therefore, both $\bar{\varphi}$ and $\tilde{\varphi}$, and hence, φ is continuous in u.

Now, let $\bar{\varphi}$ and $\tilde{\varphi}$ be population monotonic, and $\tilde{\varphi}$ be ASM. Take any $(N \cup h, \Omega, u) \in \varepsilon_2$. Note that by construction $\bar{u}_i$ is independent of the profile (the set of agents in the problem), and $\bar{u}_i = \bar{u}_i^N$ for all $i \in N$. Hence, we have $\bar{\varphi}^i(N \cup h, \Omega, \bar{u}) \leq \bar{\varphi}^i(N, \Omega, \bar{u}^N)$ for all $i \in N$ as $\bar{\varphi}$ is PM.

Let $\tilde{u}(N)$ represent the profile derived from the problem (N, Ω, u^N), and $\tilde{u}^N$ be the restricted profile of $\tilde{u}$ to the agents in N. Here, $\tilde{u}(N)$ is not necessarily equal to $\tilde{u}^N$. Define $K^0 \overset{\text{def}}{=} \{i \in N: \bar{v}(N \cup h) < u_i(\alpha\beta)\}$. Note that for $i \in N$, $\tilde{u}_i > 0$ only if $i \in K^0$. By PM (and hence SAC) we have $\tilde{\varphi}^i(N \cup h, \tilde{u}) = 0$ for all $i \in N \setminus K^0$. Hence, to complete the proof it suffices to show $\tilde{\varphi}^i(N \cup h, \tilde{u}) \leq \tilde{\varphi}^i\big(N, \tilde{u}(N)\big)$ for all $i \in K^0$.

Consider first the case $\bar{v}(N \cup h) = \bar{v}(N)$. Note that $\tilde{u}_i = \tilde{u}_i(N)$ for all $i \in N$, and we have the desired inequality as $\tilde{\varphi}$ is PM. Now, let $\bar{v}(N \cup h) - \bar{v}(N) \overset{\text{def}}{=} \Delta\bar{v} > 0$. Define $K^1 \overset{\text{def}}{=} \{i \in N: \bar{v}(N \cup h) = u_i(\alpha\beta)\}$, $K^2 \overset{\text{def}}{=} \{i \in N: 0 < \bar{v}(N \cup h) - u_i(\alpha\beta) < \Delta\bar{v}\}$, and $K = K^0 \cup K^1$. Then, $\tilde{u}_i(N) = \tilde{u}_i + \Delta\bar{v}$ for all $i \in K$, $0 < \tilde{u}_i(N) < \Delta\bar{v}$ for all $i \in K^2$, and $\tilde{u}_i(N) = 0$ otherwise. Now, consider the problem $(K, \tilde{u}^K)$. Note that PM implies the dummy property that if we add or remove agents with zero utility, allocation to the other agents does not change. Hence, we have $\tilde{\varphi}^i(N \cup h, \tilde{u}) = \tilde{\varphi}^i\big(K \cup h, \tilde{u}^{K \cup h}\big) \leq \tilde{\varphi}^i(K, \tilde{u}^K)$ for all $i \in K$, by PM. Let $\{i^1, i^2, \dots i^m\}$ be the partition of K^2 such that i^1 is the set of agents in K^2 with the highest $u_i(\alpha\beta)$, i^2 is the agents with the 2[nd] highest $u_i(\alpha\beta)$, etc. Also, define $t_0 = \bar{v}(N \cup h) - u_j(\alpha\beta)$

for $j \in i^1$, $t_p = u_j(\alpha\beta) - u_k(\alpha\beta)$ for $j \in i^p$, $k \in i^{p+1}$ and $p < m$, and $t_m = u_j(\alpha\beta) - \bar{v}(N)$ for $j \in i^m$.

Consider the problem $(K \cup i^1, \tilde{u}')$ where $\tilde{u}'_i = \tilde{u}_i + t_0$ for $i \in K$ and $\tilde{u}'_i = 0$ for $i \in i^1$. By ASM and the dummy property we have $\tilde{\varphi}^i(K \cup i^1, \tilde{u}') = \tilde{\varphi}^i(K, \tilde{u}'^K) \geq \tilde{\varphi}^i(K, \tilde{u}^K)$ for all $i \in K$. Then, add t_1 to utilities of all agents in problem $(K \cup i^1, \tilde{u}')$ and then add the agents in i^2 to the problem assigning them 0 utilities. By the same argument, no agent in K is worse off compared to $\tilde{\varphi}^i(K \cup i^1, \tilde{u}')$ and hence to $\tilde{\varphi}^i(K, \tilde{u}^K)$. Recursively applying the same argument where at the last step we add t_m to utilities of agents in $K \cup K^2$ and add the remaining agents $N \setminus (K \cup K^2)$ with zero utilities, we reach the problem $\tilde{\varphi}^i(N, \tilde{u}(N))$. As at each step, none of the agents in K gets worse off, we have $\tilde{\varphi}^i(N, \tilde{u}(N)) \geq \tilde{\varphi}^i(K, \tilde{u}^K) \geq \tilde{\varphi}^i(N \cup h, \tilde{u})$ for all $i \in K$. □

Corollary 3: The hybrid Shapley solution ($\widehat{Sh}(N, \Omega, u) = Sh(N, \Omega, \bar{u}) + Sh(N, \tilde{u})$) is efficient, symmetric, continuous and population monotonic.

The egalitarian solution is population monotonic for each component of the hybrid solution. However, the hybrid egalitarian solution is not PM.

i	$u_i(\alpha)$	$u_i(\beta)$	$u_i(\alpha\beta)$	$\bar{u}_i(\alpha\beta)$	$\tilde{u}_i(N)$	$\tilde{u}_i(\{123\})$
1	0	0	6	0	4	6
2	0	0	2	0	0	2
3	0	0	2	0	0	2
4	1	1	2	2	0	*

Table 4

Consider the problem in Table 4. Note that $\bar{v}(N) = 2$, and $\bar{v}(\{123\}) = 0$. As the egalitarian solution is in the SAC, agents 1, 2, and 3 get 0 at both problems $(N, \Omega, \bar{u})$ and $(\{123\}, \Omega, \bar{u}^{\{123\}})$. Also, $Eg^1(N, \tilde{u}) = 4$, and $Eg^1(\{123\}, \tilde{u}(\{123\})) = 2$. Thus, agent 1's final utility decreases when agent 4 leaves.

Finally, we show that ESG is not compatible with PM. Consider the problem in Table 5. For any population monotonic solution, we have $\varphi^1 + \varphi^3 \leq v(13) = 10$ and $\varphi^2 + \varphi^3 \leq v(23) = 10$. Also, $\varphi^1 + \varphi^2 + \varphi^3 = 20$. Therefore, $\varphi^3 = 0$ contradicting ESG.

i	$u_i(\alpha)$	$u_i(\beta)$	$u_i(\alpha\beta)$
1	0	10	10
2	10	0	10
3	0	0	10

Table 5

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