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Absence-proofness: A new cooperative stability concept[1]

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Abstract

We introduce a new cooperative stability concept, *absence-proofness* (AP). Given an allocation problem in a society N, and a solution well defined for all subsocieties, a group of people $S \subseteq N$ may benefit by leaving a subgroup $T \subseteq S$ "out" of the allocation process. After the allocation takes place in the society $N \setminus T$, agents in $S \setminus T$ may reallocate what they received, plus T's endowments (if they have any) among all of S. This reallocation is profitable if it is Pareto superior to what S would get in the society N had T not been left aside. We call the solutions that are immune to this kind of manipulations absence-proof. Absence-proofness implies core stability by definition. In fair division problems, where core has no bite, AP imposes core-like participation constraints on solutions. In both fair division problems and TU games, well-known *population-monotonicity* (PM) property implies AP. Although solutions that are AP but not PM exist for very specific problems, our work suggests that these properties have very close formal implications. In exchange economies with private endowments we provide many negative results. Particularly, the Walrasian allocation rule is manipulable.

Key Words: *Absence-proofness, core, stability, population-monotonicity*

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1 Introduction

Individuals involved in joint economic activities do so voluntarily; whether they show up at the scene or not depends on what they get (or expect to get) in the activity, and their outside option. If an individual or a subgroup of individuals know or anticipate that he (they) can do better on his (their) own, we expect them not to join. Designing allocation rules that make everyone or every subgroup willing to participate is the motivation for the familiar properties known as individual rationality, and core stability. We may also need to prevent the kind of partial secession discussed first in the context of exchange economies by Postlewaite (1979). Given an allocation rule, an individual may benefit by withholding some of his endowment and consuming it together with his allocation in the "reduced" economy where he provides the rest of his endowment. A rule that makes such moves unprofitable is called withholding-proof.

In the same spirit, we propose a very general group manipulation and the related concept of "*absence-proofness*" (AP). Given an allocation problem in a society N, and a solution well defined for all subsocieties, a group of people $S \subseteq N$ may benefit by leaving a subgroup $T \subseteq S$ "out" of the allocation process. After the allocation takes place in the society $N \setminus T$, agents in $S \setminus T$ may reallocate what they received, plus T's endowments (if they have any) among all of S. This reallocation is profitable if it is Pareto superior to what S would get in the society N had T not been left aside. We call the solutions that are immune to this kind of manipulations *absence-proof*.

Absence-proofness is related to both core stability and withholding-proofness (WP). Indeed, AP implies core stability, as we see by taking $T = S$. However, like WP, AP requires the knowledge of how the solution works in the problem reduced to $N \setminus T$. Therefore, while the core is defined for a single allocation problem, absence-proofness is a property of an allocation rule. Note that by taking $N = S = T$, AP also implies Pareto optimality.

Our work suggests that the absence-proofness, which is quite different than the well-known *population-monotonicity* (PM) property in spirit, has surprisingly close formal implications with PM in fair division problems and TU games. Also, AP is a very demanding property in exchange economies with private endowments.

An important feature of absence-proofness is that it applies to a larger range of models than the core and WP. Both WP and core stability apply to exchange economies or any economy where resources are privately owned to start with. Core stability also works in more abstract problems such as transferable utility (TU) cooperative games. However, both concepts lose their bite for the classical fair division problem where we distribute a commonly owned set of goods. On the other hand, AP has much to say in all these problems. Here, we study absence-proofness in three different problems; surplus sharing TU games[3], economies with private endowments, and fair division problems.

[3] Our results also apply to the cost sharing TU games with necessary adjustments where some inequalities are inversed. See Doğan (2013a) for absence-proof cost allocation rules to mcst problems.

1.1 TU games

In the TU game (N, v), AP is related to but more complicated than core stability. Agents in T can generate a surplus of $v(T)$ if they stay out. This surplus is no more than their total allocation had they appeared if a rule is a core selection (as required by AP). Therefore, for $S \supseteq T$ to manipulate, the loss of agents in T due to staying out of allocation process should be compensated by an increase in the total payoff of agents in $S \setminus T$ in the reduced problem. Thus, PM, which simply requires that no agent gains from the departure of some agents, rules out the possibility of manipulation.

Population-monotonicity has both a normative and a strategic interpretation in TU games. It was first discussed in this context by Sprumont (1990) and Moulin (1990a) (See Section 2 for a more detailed literature on PM). If an additional agent extends the cooperative opportunities, this should not harm any existing agents. On the strategic side, if an existing agent loses, he may veto the newcomer. However, analyzing this kind of strategies requires extensive modeling of the coalition formation process. The strategic move we discuss here is simple. The loser just pays the newcomer to stay out. If a rule is absence-proof, no agent i has an incentive to pay the newcomer to make him stay out even if i loses when newcomer appears.

In Section 2, we examine this one-way logical relation between PM and AP in several examples. Sönmez (1993) showed that the nucleolus is not population-monotonic on the set of convex games. In Section 2.2, we show that it does not satisfy AP, either. Sprumont (1990) showed that the bilateral assignment games[4] almost never admit a population-monotonic solution. In these games, we have a clear distinction between the two properties. In Section 2.3, we give the necessary and sufficient conditions for the existence of AP solutions in 2×2 bilateral assignment games, and propose some solutions (see Table 2.1, 2.2 and 2.3). Indeed, when they exist, absence-proof allocations constitute a central cube in the set of core allocations (see Figure 2.1). However, these conditions cannot be nicely generalized to cases with more players in each side.

In Section 2.1.1 we provide alternative definitions for the core, AP and PM. These definitions allow us to compare the three properties from a normative viewpoint. Norde and Slikker (2011) designed a set of solutions, which are general nucleoli, and population-monotonic whenever achieving PM is possible. In Section 2.4, based on our alternative definitions we reevaluate the nucleolus and design a solution ($\mathcal{K}$*-monoclus*) that is absence-proof, whenever achieving AP is possible. We also compare this solution with the nucleolus. Interestingly, in a subset of bankruptcy games they coincide.

1.2 Exchange economies with private endowments

In Section 3, we show that in the context of indivisible goods with monetary transfers, achieving AP is impossible. We illustrate this impossibility in several examples for different problems; Böhm-Bawerk markets, house assignment problems (Shapley and Shubik 1971), and

[4] Players are partitioned into two groups N_1 and N_2. The characteristic function v is defined solely by the surplus generated by the pairs $v(i, j)$, $i \in N_1$ and $j \in N_2$, and its superadditive cover.

single seller auctions. In housing markets introduced by Shapley and Scarf (1974) where no monetary compensation is available, only a weaker version of AP can be achieved. This version blocks the possibility of a strict improvement in the welfare of all agents in the manipulating coalition. We also show that the core mechanism that is calculated via the famous top trading cycle algorithm uniquely achieves the weak AP. In problems with divisible goods, on the classical domain, Walrasian allocation is not absence-proof. We also show by means of example that the manipulation of the Walrasian allocation is not a rare occurrence. Even in a problem with three agents who have the same Cobb-Douglas preferences, manipulation is possible. Our results coincide with Thomson (2013) where he introduces *withdrawal-proofness*, a concept related to but weaker than AP, in exchange economies and fair division problems. There, the manipulation argument is exactly the same. However, the manipulating coalition consists of only two agents, i.e. $|S| = 2$, and $|T| = 1$.

1.3 Fair division problems

In an environment with a fixed common endowment, it is normatively appealing to assume that no one benefits the presence of additional agents. This interpretation of PM was introduced by Thomson (1983). Here, agents have equal claims on the common endowment, and hence, the strategic interpretation of PM loses its bite. However, strategic move in the argument of AP is still intact. If agents in T renounce their claims, the manipulating coalition $S \supseteq T$ has only the allocation of $S \setminus T$ in the reduced problem to redistribute. Suppose a Pareto optimal allocation rule is population-monotonic, and S manipulates it by the absence of T. By PM, all the agents in $N \setminus S$ are better off in the reduced problem. Also, the redistribution of what $S \setminus T$ gets in the reduced problem to all the agents in S is Pareto superior to the allocation to S in the society N. This contradicts that the allocation in society N is Pareto optimal. Hence, under efficiency, PM implies AP (Theorem 4.1). For a related literature on PM, and the corollaries see Section 4.2 and 4.3.

Aside from this strong sufficient condition (PM) for AP, we provide a sufficient condition for manipulation (see Proposition 4.2). As a corollary, in Section 4.2, we show that the competitive equilibrium with equal incomes solution is not AP in the problem of distributing a perfectly divisible bundle in $\mathbb{R}^l_+$.

The simplicity of the model of allocating a single object with monetary transfers enables us to give a simple characterization of AP rules, which clarifies the difference between PM and AP (see Propositions 4.3 and 4.5). Indeed, it shows how close they are. Thus, by replacing PM with AP, we cannot escape the incompatibility between one of the most desired fairness property “envy-freeness” (Foley 1967) and PM (see Alkan (1994) and Moulin (1990b)). However, rules that are AP but not PM exist and we introduce two of them. First one is a serial oligarchy solution. The other lexicographically favors the agents, starting with the agent who has the lowest valuation for the object while respecting two upper bounds (see Proposition 4.7). A weaker version of PM requires that when some agents leave, remaining agents should be affected in the same direction. The second rule we introduce is not even weakly PM.

2 Absence-proofness in TU Surplus Sharing Games

Basic notions Given a society $N = \{1,2,\dots,n\} \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of all finite societies, a characteristic function $v: 2^N \to \mathbb{R}_+$ describes the value that a group of agents $S \subseteq N$ are able to create on their own, with the convention that $v(\emptyset) = 0$. Hence, a tuple (N, v) defines a *surplus sharing TU game*.

For a fixed (N, v), an *allocation* $x = (x_i)_{i \in N} \in \mathbb{R}_+^N$ is a vector s.t. $\sum_{i \in N} x_i \leq v(N)$. x is *efficient* if $\sum_{i \in N} x_i = v(N)$, and *individually rational* if $x_i \geq v(i)$ for all $i \in N$. The *imputation set* $I(N, v)$ consists of all the efficient and individually rational allocations. Given (N, v) and $S \subseteq N$, the game (S, v_S) is a *subgame* of (N, v) if for all $T \subseteq S$, $v_S(T) = v(T)$. By abuse of notation, we will write (S, v) instead of (S, v_S) or even sometimes game S when there is no confusion about the fixed (N, v). An *allocation scheme* $\mathcal{X}(\cdot)$ is a mapping that assigns an efficient allocation to each subgame, i.e. $\mathcal{X}(S) \in \mathbb{R}_+^S$ for all $S \subseteq N$ and $\sum_{i \in S} \mathcal{X}_i(S) = v(S)$.[5]

Let Γ^N be the set of all games admissible for N and $\Gamma = \bigcup_{N \in \mathcal{N}} \Gamma^N$. An *allocation rule* defined on the domain of games $\mathcal{G} \subseteq \Gamma$ assigns an allocation at each game in $\mathcal{G}$. We call a domain $\mathcal{G}$ *rich* if for any game (N, v), all the subgames of that game are in $\mathcal{G}$, as well. Note that an allocation rule on a rich domain $\mathcal{G}$ induces a unique allocation scheme for each game in $\mathcal{G}$ but not vice versa. A game (N, v) is *convex* if for all $T \subseteq S \subseteq N$ and $i \notin S$, we have $v(T \cup \{i\}) - v(T) \leq v(S \cup \{i\}) - v(S)$.

The *core* of a game (N, v) is the set $C(N, v) = \{x \in \mathbb{R}_+^N : \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N\}$. A game (N, v) is *balanced* if it has a nonempty core, and *totally balanced* if all of its subgames are balanced. An allocation scheme is a *core selection* if it assigns a core allocation to each subgame. Similarly, an allocation rule is a core selection on $\mathcal{G}$ if it assigns a core allocation to each game in $\mathcal{G}$.

2.1 Absence-proof allocation schemes

Core stability is the most fundamental stability property in the context of TU games. If not satisfied, a group of agents T would stay out and enjoy the surplus of $v(T)$, which is more than their total payoff. This group decision requires three basic assumptions. Agents should know or anticipate the outcome, act voluntarily, and surplus generation as well as the redistributions out of the allocation process is possible. Without any further assumptions, by the help of following example, we discuss a new outside option that is neither foreseen, nor prevented by the core stability.

Example 2.1: $N = \{1, \dots, 5\}, W = \{1,2,3\}$ and $F = \{4,5\}$; $v(S) = \min\{|S \cap W|, |S \cap F|\}$, for all $S \subseteq N$.

There are three workers with the same skill, and two firms are looking for exactly one worker with that skill. Each firm can employ at most one worker. A worker can create 1 unit of surplus if hired, and is useless otherwise. Here, the core is simple to describe. For each

[5] By abuse of notation, we write $\mathcal{X}(S)$ instead of $\mathcal{X}(S, v)$, and $\mathcal{X}(1,2, \dots, s)$ instead of $\mathcal{X}(\{1,2, \dots, s\})$.

subgame S, as long as $|S \cap W| \neq |S \cap F|$, the scarce type of agents in S equally share the entire surplus of $v(S)$. Hence, the unique core allocations of game N and subgame $S = \{1,4,5\}$ are $x = (0,0,0,1,1)$, and $x' = (1,*,*,0,0)$, respectively. It is also well known that the game (N, v) in the example is totally balanced.

Let $\mathcal{X}(\cdot)$ be a core selection. Then, $\mathcal{X}(N) = (0,0,0,1,1)$ and $\mathcal{X}(1,4,5) = (1,*,*,0,0)$. Consider the coalition of workers W. Agent 1 asks the other two to stay at home. Then, the active agents are $1, 4$ and 5. As two firms are competing to hire the same worker, agent 1 gets the entire surplus of 1 unit in game $\{1,4,5\}$. Note that any redistribution of 1 unit among the workers is a Pareto improvement with respect to their allocation in $\mathcal{X}(N)$.

Definition 2.1: Let (N, v) be given, $\mathcal{X}$ is an *absence-proof allocation scheme* (*APAS*) if for all $T \subseteq K \subseteq S \subseteq N$,

$$\textstyle\sum_{i \in K \setminus T} \mathcal{X}_i\,(S \setminus T) + v(T) \leq \sum_{i \in K} \mathcal{X}_i\,(S) \qquad (1)$$

Here, we enlarge the outside options of the coalition of K. In addition to a total secession of the coalition (the case $K = T$ in (1)), as prevented by core stability, K may also leave a strict subset of it outside the allocation process, and can still benefit. These additional options correspond to the case $T \subset K$ in (1).

When we think of N as a maximal society and the population as a variable, the domain $\mathcal{G}$ of games we would face are (N, v) and all of its subgames. Hence, allocation schemes are allocation rules defined on $\mathcal{G}$. Adopting this interpretation, we ask $\mathcal{X}(\cdot)$ to satisfy the desired property not only in game (N, v), but in all the subgames , i.e. for all $S \subseteq N$.[6]

Remark 2.1: An APAS is core selection, and hence efficiency at problem N even had we not imposed efficiency on $\mathcal{X}(\cdot)$ by definition. Just set $K = T$.

In TU game context, allocation schemes are widely used solution objects, especially in the literature on the well-known population-monotonicity property. The same object appears as generalized allocation in Moulin (1990a), and payoff configuration in Thomson (1995). However, use of allocation schemes is more common, following Sprumont (1990) where he defines the *population-monotonic allocation schemes* (*PMAS*). An allocation scheme $\mathcal{X}(\cdot)$ is PMAS if $\mathcal{X}_i(T) \leq \mathcal{X}_i(S)$ for all $T \subseteq S \subseteq N$, $i \in T$. Notice a PMAS $\mathcal{X}(\cdot)$ is a core selection, as $v(T) = \sum_{i \in T} \mathcal{X}_i\,(T) \leq \sum_{i \in T} \mathcal{X}_i\,(S)$ for all $T \subseteq S \subseteq N$.

The game in Example 2.1 clearly does not admit an APAS. Note that, there, the payoff of worker 1 increases by absence of the other workers. Indeed, this is not a coincidence, but a requirement for the workers to manipulate the allocation scheme.

Sprumont (1990) is the first to discuss the strategic interpretation of PM. He argues that if an agent i gets more in a subgame, say $\mathcal{X}_i(N) < \mathcal{X}_i(S)$, then i will be tempted to form the smaller coalition S by using his bargaining skills or by any other means. The strategic move we define here is indeed a particular action that i could take; that is convincing $N \setminus S$ to stay out by paying them.

[6] See Sprumont (2008) for a detailed argument on the use of allocation schemes.

Proposition 2.1: Given a game (N, v), if $\mathcal{X}$ is a PMAS, it is also an APAS.

Proof: Let (N, v) be a game and $\mathcal{X}$ be a PMAS at this game. Take any $T \subseteq K \subseteq S \subseteq N$. PM implies $\sum_{i \in K \setminus T} \mathcal{X}_i\,(S \setminus T) \leq \sum_{i \in K \setminus T} \mathcal{X}_i\,(S)$, and $v(T) = \sum_{i \in T} \mathcal{X}_i\,(T) \leq \sum_{i \in T} \mathcal{X}_i\,(S)$. Then, (1) holds. □

Proposition 2.1 provides a strong reason to choose PMAS's among the core selections. It also tells us a lot about APAS's. First of all any game that admits a PMAS also admits an APAS. Any game that is a linear combination of monotonic simple games, and only those, admits a PMAS (Sprumont 1990). Norde and Reijnierse (2002) give another characterization by generalizing the idea of vector of balanced weights that is used to characterize the balanced games. Existence of (maybe a similar) a nice and compact characterization of the set of games that admit an APAS is still an open question. We provide a characterization for 2×2 bilateral assignment games in Section 2.3.

Other corollaries to Proposition 2.1 are as follows: The Shapley value (Shapley 1962) and sequential and monotone Dutta-Ray solutions (Dutta and Ray 1989) are absence-proof on the set of convex games (Sprumont 1990 and Hokari 2002). The proportional allocation scheme is absence-proof on average monotonic games (Izquierdo and Rafels 2001).

By Remark 2.1, AP solutions exist only if the game is totally balanced. We know by Sönmez (1993) that the nucleolus (Schmeidler 1969) is not PM on convex games. In Section 2.2, we show that it is not even AP. Sprumont (1990) and Innara (1993) proved that the Shapley value is a core selection on the class of average (quasi) convex games[7] while Sprumont also showed that the Shapley value is not a PMAS on this domain.

Open Question 1: Is the Shapley value AP on the set of average (quasi) convex games?

It is important to keep in mind that the formulation in (1) is critical and not appropriate for all problems that are represented by a TU game. TU games are commonly simplified representations of the feasible utility space in allocation problems with private or common endowments, and quasilinear preferences that allow monetary transfers. In a fair division problem, for all $T \subseteq N$, $v(T)$ represents the monetary equivalent of the common endowment for agents in T when they are the only claimants. However, when the set of claimants is $N \supset T$, coalition T can generate no surplus if they renounce their claim and leave the scene. Inequality (1) does not represent this situation. The argument is more subtle for the case of private endowments. Suppose we are reallocating privately owned indivisible goods, say cars, and T stays out with the cars they own. $K \setminus T$ brings its allocation after the allocation process in the society $N \setminus T$. If an agent from T has a utility higher than everyone else for a car that $K \setminus T$ brought in, he gets it in the Pareto improving reallocation. Then, the monetary equivalent of K's "new" total endowment (cars that $K \setminus T$ got in the allocation process plus the cars that agents in T own) for agents in K is more than $\sum_{i \in K \setminus T} \mathcal{X}_i\,(N \setminus T) + v(T)$. So, the results in this section are valid for TU games where the allocation is in actual money terms.

[7] A game (N, v) is quasi-convex if for all $S \subseteq T \subseteq N$, we have $\sum_{i \in S}\big(v(S) - v(S \setminus i)\big) \leq \sum_{i \in S}\big(v(T) - v(T \setminus i)\big)$

As an end note for the formulation (1), we may have asked for a weaker property which ensures no manipulation only at the grand game (N, v), and hence (1) to hold only for $S = N$. In that case, for any balanced game we can write a "weakly" absence-proof allocation scheme. It is an easy exercise to see that (1) holds for the case $S = N$ for the following "proportional" allocation scheme: Fix an arbitrary $y \in C(N, v)$. Define for all $i \in N$, $\mathcal{X}_i(N) = y_i$; for all $S \subseteq N, i \in S$, $\mathcal{X}_i(S) = y_i(v(S)/y(S))$ if $y(S) > 0$, and $\mathcal{X}_i(S) = 0$ if $y(S) = 0$.

Applying the proportional allocation scheme to the game in Example 2.1, we have $\mathcal{X}(N) = (0,0,0,1,1)$, and $\mathcal{X}(1,4,5) = (0,*,*,0.5,0.5)$. Then, if for some reason workers 2 and 3 are not available anymore, it does not assigns a core allocation to the game $\{1,4,5\}$.

2.1.1 Comparing the core, AP and PM: a normative approach

Besides being a stability property, core can be conceived as a normative property as well. Let (N, v) be a totally balanced game and $\mathcal{X}$ be a core selection. Consider two disjoint subsocieties S, S' and the subgame $S \cup S'$. Note that for any totally balanced game, we can write $v(S \cup S') = v(S) + v(S') + g(S, S')$ with $g(S, S') \geq 0$. Both subsocieties guarantee a payoff of $v(S)$ and $v(S')$, respectively at $\mathcal{X}$ in subgame $S \cup S'$. Then, the remaining $g(S, S')$ is distributed among the agents in $S \cup S'$. So, if we interpret $g(S, S')$ as the value created by merger of two societies, core dictates that none of the two societies should gain in total more than that amount when they merge. The next proposition relates this normative interpretation to absence-proofness, and population-monotonicity.

Proposition 2.2: Let (N, v) be a game, $\mathcal{X}$ be an allocation scheme and $S, S' \subseteq N$ such that $S \cap S' = \emptyset$.

(i) $\mathcal{X}$ is a core selection if and only if we have,

$$\sum_{i \in S}\big(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S)\big) \leq g(S, S') \tag{2}$$

(ii) $\mathcal{X}$ is an APAS if and only if for all $K \subseteq S$ we have,

$$\sum_{i \in K}\big(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S)\big) \leq g(S, S') \tag{3}$$

(iii) $\mathcal{X}$ is a PMAS if and only if for all $K \subseteq S \cup S'$ we have,

$$\sum_{i \in (K \cap S)}\big(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S)\big) + \sum_{i \in (K \cap S')}\big(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S')\big) \leq g(S, S') \tag{4}$$

Proof: We only prove (ii). and omit the trivial arguments for (i). and (iii).

Necessity. Let $\mathcal{X}$ be an APAS at (N, v). Take any $S, S' \subseteq N$ s.t. $S \cap S' = \emptyset$, and any $K \subseteq S$. (1) implies that $\sum_{i \in S \setminus K} \mathcal{X}_i(S) + v(S') \leq \sum_{i \in (S \setminus K) \cup S'} \mathcal{X}_i(S \cup S')$. Then, $v(S) - \sum_{i \in K} \mathcal{X}_i(S) + v(S') \leq v(S \cup S') - \sum_{i \in K} \mathcal{X}_i(S \cup S')$. Therefore, (3) is satisfied.

Sufficiency. Let (N, v) be a game, $\mathcal{X}$ be an allocation scheme at (N, v), and (3) hold. Take any $T \subseteq K \subseteq S \subseteq N$. Then, $(S \setminus K) \subseteq (S \setminus T)$, and by (3), $\sum_{i \in S \setminus K} \mathcal{X}_i(S) - \sum_{i \in S \setminus K} \mathcal{X}_i(S \setminus T) \leq v(S) - v(S \setminus T) - v(T) = g(S \setminus T, T)$. Thus, $v(S \setminus T) - \sum_{i \in S \setminus K} \mathcal{X}_i(S \setminus T) + v(T) \leq v(S) - \sum_{i \in S \setminus K} \mathcal{X}_i(S)$. Therefore, $\sum_{i \in K \setminus T} \mathcal{X}_i(S \setminus T) + v(T) \leq \sum_{i \in K} \mathcal{X}_i(S)$, and hence (1) holds. □

By Proposition 2.2, we can read absence-proofness as follows: When two societies merge, no coalition from one of these societies gets an extra surplus that is more than the value created ($g(S, S')$) by this merger, while population-monotonicity does not allow any coalitions from the joint society to gain more than this value.

2.2 Nucleolus on the set of convex games

The class of convex games has been a special area of interest in the literature on PMAS for two reasons: they constitute a rich domain, and usually have a large set of core allocations. The literature on PM mainly focused on allocation schemes that are based on applying allocation rules (or sometimes referred to as value operators) to all subgames of a game in a given rich domain. Sönmez (1993) showed that the "extended" nucleolus is not PM on convex games in general. However, on a particular subset of (dual of) these games, known as airport games, it is PM.

The lexicographic ordering of $\mathbb{R}^l$ is denoted by $\succcurlyeq_L$; that is $x \succcurlyeq_L y$ for $x, y \in \mathbb{R}^l$ if $x = y$ or there is $t \in \{1, \dots, l\}$ such that $x_{t'} = y_{t'}$ for all $t' < t$ and $x_t > y_t$.

Now, let (N, v) be such that $I(N, v)$ is nonempty. For each allocation $x \in I(N, v)$ define the excess of coalition $S \subseteq N$ as $e(x; S) = \sum_{i \in S} x_i - v(S)$. Let $e(x) \in \mathbb{R}^{|2^N|}$ have the excesses of allocation x ordered increasingly. Then, the unique allocation $\mu(N, v)$ such that for each $x \in I(N, v)$, $e\big(\mu(N, v)\big) \succcurlyeq_L e(x)$ is called the *nucleolus* of the game. Let $\mathcal{M}$ denote the extended nucleolus, i.e. given any game (N, v), $\mathcal{M}(S) = \mu(S, v)$ for all $S \subseteq N$.

Proposition 2.3: The extended nucleolus $\mathcal{M}$ is not AP on the set of convex games.

Proof: Consider the following games (N, v), and (N', v'); $N = \{1, \dots, 6\}$, $N' = N \cup \{7\}$:

$$v(S) = \begin{cases} 3 & \text{if } |S| = 3 \text{ and } 6 \in S \\ 6 & \text{if } |S| = 4 \text{ and } 6 \in S \\ 9 & \text{if } |S| = 5 \text{ and } 6 \in S \\ 12 & \text{if } \ S = N \\ 0 & \text{otherwise} \end{cases} \qquad v'(S) = \begin{cases} v(S) & \text{if } 7 \notin S \\ v(S \setminus \{7\}) & \text{if } 7 \in S \\ 12.4 & \text{if } S = N \end{cases}$$

Note that (N, v) is a subgame of (N', v') and both games are convex. Also, for all $i, j \in K \stackrel{\text{def}}{=} \{1,2,3,4,5\}$ and S s.t. $i, j \notin S$ we have $v(S \cup i) = v(S \cup j)$, and $v'(S \cup i) = v'(S \cup j)$. It is easy to check that here nucleolus treats these agents equally, and we have $\mu_i(N) = y$ for all $i \in K$ and $\mu_6(N) = 12 - 5y$. Then, the minimum excess is maximized for $S = \{i\}$ with $i \in K$ and $y = 1.5$. Hence we have;

$$\mu_i(N) = 1.5 \text{ for all } i \in K \text{ and } \mu_6(N) = 4.5$$

In game (N', v'), the minimum excess is trivially maximized for $S = \{7\}$ and $S' = N' \setminus \{7\}$. Then, we have $\mu_7(N') = 0.2$ and $\sum_{i \in N} \mu_i(N') = 12.2$. The argument for distributing 12.2 to agents in N is similar to the one in game (N, v). Hence, we have;

$$\mu_i(N') = 1.6 \text{ for all } i \in K;\ \mu_6(N') = 4.2 \text{ and } \mu_7(N') = 0.2$$

Then, by absence of agent 7 at game (N', v'), agents 6 and 7 would enjoy a total of 4.5 instead of 4.4. □

2.3 Bilateral assignment games

The society N consists of two disjoint type of agents A and B, i.e. $N = A \cup B$, and $A \cap B = \emptyset$. No coalition consisting of agents only from A or B can create a surplus. A generic pair $(a_i, b_j) \in A \times B$ can create $v(i,j) \geq 0$. A coalition S containing several agents of each type generates the surplus $v(S)$ by forming pairs (a_i, b_j) efficiently and summing up the corresponding $v(i,j)$'s, i.e. for $S = A' \cup B'$ s.t. $A' \subseteq A$ and $B' \subseteq B$, $v(S)$ is the maximal sum $\sum v(i,j)$ when we assign agents of A' to those of B'. We call an assignment of A' to B' optimal if it generates $v(S)$, and we say that a pair is optimal in S if it appears in some optimal assignment that generates $v(S)$. We will represent a bilateral assignment game by the matrix $v(A \times B)$.

Regarding the economic environments that would induce this type of games, we can think of A as the set of potential workers with similar skills, and B as the firms that needs only one worker with this specific skill. We can also think of A as a set of flute players, and B as a set of piano players, who are seeking performance positions for duos.

One nice feature of these games is that they are totally balanced. However, as we show in Proposition 2.5, a PMAS is almost never admissible for this class of games. Although, there are games which admit an APAS, there are severe limitations on the surplus opportunities. In games with two agents on each side we are able to identify the necessary and sufficient conditions for the existence of an APAS. Unfortunately, there is no simple procedure to generate an APAS. Moreover, these conditions would not be nicely generalized to the games with $n \geq 5$.

Lemma 2.1: *Shapley and Shubik (1971).* Given a bilateral assignment game $((A;B), v)$, $x(N)$ is a core allocation if and only if for all $i \in A$, $j \in B$ we have $x_i(N) + x_j(N) \geq v(i,j)$; with equality if (a_i, b_j) is an optimal pair in N. An agent who is not in any optimal pair gets 0.

Lemma 2.2: Let $((A;B), v)$ be an assignment game, and (a_i, b_j) be an optimal pair at subgame (S, v). Then, for any APAS $\mathcal{X}$ we have, $\mathcal{X}_i(S) = \mathcal{X}_i(i,j)$, and $\mathcal{X}_j(S) = \mathcal{X}_j(i,j)$.

Proof: Let everything be as in the statement of the lemma. By Lemma 2.1, we have $\mathcal{X}_i(S) + \mathcal{X}_j(S) = \mathcal{X}_i(i,j) + \mathcal{X}_j(i,j) = v(i,j)$. Also, as (a_i, b_j) is an optimal pair at (S, v), $v(i,j) + v(S \backslash \{i,j\}) = v(S)$. Let wlog $\mathcal{X}_i(S) < \mathcal{X}_i(i,j)$. Then, as $\mathcal{X}_i(i,j) + v(S \backslash \{i,j\}) > \mathcal{X}_i(S) + \sum_{k \in S \backslash \{i,j\}} \mathcal{X}_k(S)$, the coalition $S \backslash \{j\}$ is better off by leaving $S \backslash \{i,j\}$ out of the game S. □

The idea in Lemma 2.2 is crucial for the characterization of APAS's in 3-person games, and the conditions for the existence of an APAS in 4-person (2×2) games. As the size of the society grows, in case (a_i, b_j) is an optimal pair in game N, it is possibly optimal for many other subgames of N. Lemma 2.2 indicates that absence-proofness become more restrictive with the size of N.

A 3-person (1×2) game is defined by two numbers and a 4-person (2×2) game is defined by four numbers. To get rid of notational complexity we write these numbers in an increasing manner i.e., $u_1 \leq u_2$ for 3-person games and $u_1 \leq u_2 \leq u_3 \leq u_4$ for 4-person games.

Proposition 2.4: Let $((A;B),v)$ be a generic 3-person game as shown in the matrix. Then, $\mathcal{X}$ is an APAS if and only if it is efficient at each S, $\mathcal{X}_1(N) = \mathcal{X}_1(1,2) \geq u_1$, $\mathcal{X}_2(N) = \mathcal{X}_2(1,2)$, $\mathcal{X}_3(N) = 0$ and $\mathcal{X}_2(N) \geq \mathcal{X}_3(1,3)$.

	b_2	b_3
a_1	u_2	u_1

Here in 3-person games, the only difference between an APAS and a PMAS is that a PMAS gives nothing to agent 3 in games N and $\{1,3\}$, while an APAS admits a positive payoff to agent 3 in game $\{1,3\}$. For the sake of completeness we now duplicate the result, Proposition 2 in Sprumont (1990).

Proposition 2.5: Let $((A;B),v)$ be a bilateral assignment game with $|A|,|B| \geq 2$ such that for some $i,i' \in A$ and $j,j' \in B$, we have $v(k,l) > 0$ for $k \in \{i,i'\}$, and $l \in \{j,j'\}$. Then, game $((A;B),v)$ does not admit a PMAS.

Proof: Let $((A;B),v)$ be as in the premise of the proposition with $\{i,i'\} = \{1,2\}$, $\{j,j'\} = \{3,4\}$ and wlog $v(1,3) = u_1$. Suppose for a contradiction that $\mathcal{X}$ is a PMAS. Then, $\mathcal{X}$ is a core selection, and by Lemma 2.1 we have, $\mathcal{X}_1(1,2,3) = \mathcal{X}_3(1,3,4) = 0$. Then, PM implies $\mathcal{X}_1(1,3) = \mathcal{X}_3(1,3) = 0$ and this contradicts that $\mathcal{X}$ is efficient. □

Proposition 2.6: Let $((A;B),v)$ be a game s.t. $|A| = |B| = 2$. There are three generic configurations of the value matrix and the conditions stated for each generic case are necessary and sufficient for the existence of an APAS.

<u>Case 1:</u> u_3, u_4 are diagonal

	b_3	b_4
a_1	u_1	u_4
a_2	u_3	u_2

$\Rightarrow$ $(i)\ u_3 \geq u_1 + u_2$

$(ii)\ u_3 + u_4 \geq 3u_2$

<u>Case 2.1:</u> u_3, u_4 are inline

	b_3	b_4
a_1	u_1	u_3
a_2	u_2	u_4

$\Rightarrow$ $(i)\ u_4 \geq u_2 + u_3$

$(ii)\ u_2 + u_3 \geq 3u_1$

<u>Case 2.2:</u> u_3, u_4 are inline

	b_3	b_4
a_1	u_1	u_4
a_2	u_2	u_3

$\Rightarrow$ $(i)\ u_3 \geq u_1 + u_2$

$(ii)\ u_4 \geq u_2 + u_3$

Note that the game in Example 2.1 is a particular bilateral assignment game where all entries are 1. Any 2×2 subgame violates all the conditions above. In Appendix, we show that the allocation scheme in Table 2.1 is an APAS for Case 1. We also propose two different APAS's for the other cases, and show that the conditions in each case are necessary.

When all entries are strictly positive, for Case 1 and Case 2.2, efficiency induces a unique optimal assignment. For Case 2.1, in case $u_1 > 0$, absence-proofness requires a unique optimal assignment where agents who generate the highest payoff of u_4 are matched.

$\boldsymbol{S}$	$\mathcal{X}_1(\boldsymbol{S})$	$\mathcal{X}_2(\boldsymbol{S})$	$\mathcal{X}_3(\boldsymbol{S})$	$\mathcal{X}_4(\boldsymbol{S})$
$\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$	$u_4 - u_2$	u_2	$u_3 - u_2$	u_2
$\{\mathbf{1},\mathbf{2},\mathbf{3}\}$	0	u_2	$u_3 - u_2$	$*$
$\{\mathbf{1},\mathbf{2},\mathbf{4}\}$	$u_4 - u_2$	0	$*$	u_2
$\{\mathbf{1},\mathbf{3},\mathbf{4}\}$	$u_4 - u_2$	$*$	0	u_2
$\{\mathbf{2},\mathbf{3},\mathbf{4}\}$	$*$	u_2	$u_3 - u_2$	0
$\{\mathbf{1},\mathbf{3}\}$	$u_1/2$	$*$	$u_1/2$	$*$
$\{\mathbf{1},\mathbf{4}\}$	$u_4 - u_2$	$*$	$*$	u_2
$\{\mathbf{2},\mathbf{3}\}$	$*$	u_2	$u_3 - u_2$	$*$
$\{\mathbf{2},\mathbf{4}\}$	$*$	$u_2 - \mathcal{X}_4(2,4)$	$*$	$\min\{u_3 - u_2, u_2/2\}$

Table 2.1

Even in a 2×2 game, AP rules out most of the core allocations available at (N, v). Consider the game $(N, v) = \big((A; B), v\big)$ in Case 1 of Proposition 2.6. Let $(\alpha_1, \alpha_2, \beta_3, \beta_4)$ be an efficient allocation. By Lemma 2.1, core is described by the following inequalities:

$$\alpha_1 + \beta_4 = u_4 \,,\, \alpha_2 + \beta_3 = u_3 \,,\, \alpha_1 + \beta_3 \geq u_1 \,,\, \alpha_2 + \beta_4 \geq u_2$$

We can rewrite these inequalities as follows:

$$\alpha_1 + (u_3 - u_1) \geq \alpha_2 \,,\, \alpha_2 + (u_4 - u_2) \geq \alpha_1 \,,\, u_4 \geq \alpha_1 \,,\, u_3 \geq \alpha_2$$

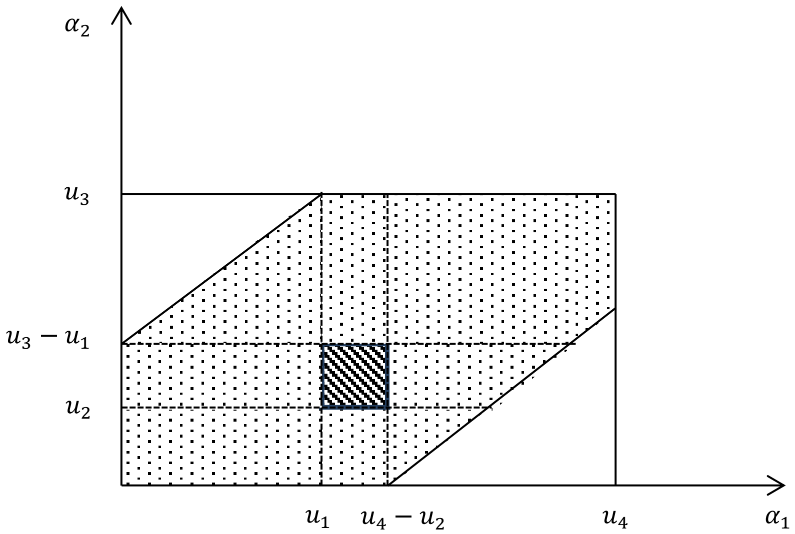

Figure 2.1

The dotted area in Figure 2.1 shows the set of all core allocations. By Lemma 2.2 agents 1 and 4 gets (α_1, β_4) in games $\{1,2,4\}$, and $\{1,3,4\}$. Then, applying Proposition 2.4 to these games, we have the first inequality below. By a similar argument for agents $\{2,3\}$, and games

$\{1,2,3\}$ and $\{2,3,4\}$, we have the second inequality. So, AP induces the following additional restrictions on this allocation:

$$u_4 - u_2 \geq \alpha_1 \geq u_1 \text{ , } u_3 - u_1 \geq \alpha_2 \geq u_2$$

The dashed area in Figure 2.1 shows the set of core allocations that an APAS may assign to game (N, v). This rules out the two famous (extreme) solutions, A-optimal and B-optimal allocations. Here, the A-optimal allocation is $\alpha_1 = u_4$, $\alpha_2 = u_3$ and conversely B-optimal allocation is $\alpha_1 = \alpha_2 = 0$. The allocation scheme in Table 2.1 corresponds to the southeast corner of the dashed square below. We can think of the northeast and the southwest corners of that square as the restricted A-optimal and B-optimal allocations, respectively. Unfortunately, we may not be able to write an APAS that assigns one of those two allocations at (N, v) regardless of the surplus opportunities, even though they satisfy both conditions in Case 1. We omit the exhaustive argument.

Note that the boundaries of the square in Figure 2.1 are derived by Proposition 2.4, while the conditions in Proposition 2.6 ensure that this set is nonempty. The following example shows that even if all the 2×2 subgames of a game satisfy the conditions in Proposition 2.6, that game may not admit an APAS. The game in the example is only a 2×3 game, and this gives us a hint of how much more restrictive absence-proofness becomes as the society expands.

Example 2.2:

	b_3	b_4	b_5
a_1	3	5	0
a_2	2	8	3

Note that, all three of the 2×2 subgames fall into a different case of the Proposition 2.6. Suppose $\mathcal{X}$ is an APAS at the game (N, v). The optimal assignment with the surplus of 11 is $\big((a_1, b_3), (a_2, b_4)\big)$. By Proposition 2.4, we have $\mathcal{X}_3(1,2,3) = \mathcal{X}_3(1,3) \geq 2$, and $\mathcal{X}_4(1,2,4) = \mathcal{X}_4(2,4) \geq 5$. By Lemma 2.2, we have $\mathcal{X}_3(N) \geq 2$, and $\mathcal{X}_4(N) \geq 5$. Thus, $\mathcal{X}_1(N) + \mathcal{X}_2(N) + \mathcal{X}_5(N) \leq 4$. Now, let $T = \{2,5\}$ and consider the game $N \setminus T = \{1,3,4\}$. By Proposition 2.4, $\mathcal{X}_1(1,3,4) \geq 3$. Then, as $v(T) = 3$, $\{1,2,5\}$ would be better off by absence of T at (N, v). Thus, this game does not admit an APAS.

2.4 General nucleoli

In the literature on PMAS, the primary issue has been to figure out if well-known allocation rules satisfy population-monotonicity on a certain class of games. Norde, and Slikker (2011) departed this trend and focused on constructing PMAS's for an arbitrary game (N, v). They introduced the solution *monoclus* (a general nucleolus, see Maschler et al (1992)) and some variations, which are population-monotonic whenever a PMAS exists. Among those solutions, we inspire from $\mathcal{T}$-monoclus and define $\mathcal{K}$-monoclus, which is absence-proof whenever an APAS exists.

Given a game (N, v) and an allocation scheme $\mathcal{X}$, for any triple (i, S, T) with $i \in S \subset T \subseteq N$, the *monotonicity* of $\mathcal{X}$ with respect to (i, S, T) is defined as $mon\big(\mathcal{X}, (i, S, T)\big) = \mathcal{X}_i(T) -$

$\mathcal{X}_i(S)$. Note that an allocation scheme is population-monotonic if and only if all the monotonicities are nonnegative.

Now, define the set $\mathcal{T} = \{(i, S, T): i \in S \subset T \subseteq N\}$ and let $\theta(\mathcal{X}) \in \mathbb{R}^{|\mathcal{T}|}$ have all the monotonicities of $\mathcal{X}$ corresponding to elements in $\mathcal{T}$ as its coordinates, in a weakly increasing order. Let $\mathbb{X}(I(N, v))$ denote the set of all allocation schemes at game (N, v) that yields efficient and individually rational allocations at each subgame i.e., $\mathcal{X}(S, v) \in I(S, v)$ for all $S \subseteq N$ for all $\mathcal{X} \in \mathbb{X}(I(N, v))$. Then, $\mathcal{T}$-*monoclus* is defined as follows:

$$\mathcal{M}^{\mathcal{T}}(N, v) = \{\mathcal{X} \in \mathbb{X}(I(N, v)): \theta(\mathcal{X}) \succcurlyeq_L \theta(Y) \text{ for all } Y \in \mathbb{X}(I(N, v))\} \tag{5}$$

$\mathcal{M}^{\mathcal{T}}(N, v)$ is nonempty and consist of only one element at each game with a nonempty imputation set $I(N, v)$ as the set $\{\theta(\mathcal{X}) \in \mathbb{R}^{|\mathcal{T}|}: \mathcal{X} \in \mathbb{X}(I(N, v))\}$ is convex and compact. Note that if a game admits an allocation scheme $\mathcal{X}$ with a nonnegative $\theta(\mathcal{X})$, and hence admits a PMAS, $\theta\left(\mathcal{M}^{\mathcal{T}}(N, v)\right)$ is also nonnegative, and $\mathcal{M}^{\mathcal{T}}(N, v)$ is a PMAS.

2.4.1 The extended nucleolus

By Proposition 2.2, we say an allocation scheme is a core selection if and only if we have $g(P, Q \setminus P) - \sum_{i \in P}(\mathcal{X}_i(Q) - \mathcal{X}_i(P)) \geq 0$ for all $P \subset Q \subseteq N$. We can rewrite the inequality as follows:

$$v(Q) - v(P) - v(Q \setminus P) - \textstyle\sum_{i \in P} \mathcal{X}_i(Q) + v(P) \geq 0 \Leftrightarrow \sum_{i \in Q \setminus P} \mathcal{X}_i(Q) - v(Q \setminus P) \geq 0.$$

Renaming $Q \setminus P = S$, and $Q = T$, we have $\sum_{i \in S} \mathcal{X}_i(T) - v(S) \geq 0$ for all $S \subset T \subseteq N$. Now, define $\Sigma mon(\mathcal{X}, (S, T)) = \sum_{i \in S} \mathcal{X}_i(T) - v(S)$, and the set $\mathcal{S} = \{(S, T): S \subset T \subseteq N\}$. Note that $\Sigma mon(\mathcal{X}, (S, T)) = \sum_{i \in S} mon(\mathcal{X}, (i, S, T)) = e(\mathcal{X}(T); S)$ (as defined in Section 2.2). Let $\varepsilon(\mathcal{X}) \in \mathbb{R}^{|\mathcal{S}|}$ be the vector that has all the Σ-monotonicities of $\mathcal{X}$ corresponding to elements in $\mathcal{S}$ as its coordinates in a weakly increasing order. Then, $\mathcal{S}$-*monoclus* is as follows:

$$\mathcal{M}^{\mathcal{S}}(N, v) = \{\mathcal{X} \in \mathbb{X}(I(N, v)): \varepsilon(\mathcal{X}) \succcurlyeq_L \varepsilon(Y) \text{ for all } Y \in \mathbb{X}(I(N, v))\} \tag{6}$$

As the set $\varepsilon \stackrel{\text{def}}{=} \{\varepsilon(\mathcal{X}) \in \mathbb{R}^{|\mathcal{S}|}: \mathcal{X} \in \mathbb{X}(I(N, v))\}$ is convex and compact, $\mathcal{M}^{\mathcal{S}}(N, v)$ that lexicographically maximizes $\varepsilon(\mathcal{X})$ within ε exists and is unique at each game with a nonempty imputation set $I(N, v)$.

Note that $\varepsilon(\mathcal{X})$ consist of the excess vectors $e(\mathcal{X}(T); S)$ associated with the allocation $\mathcal{X}(T)$ induced by $\mathcal{X}$ at each subgame T. The difference between the $\mathcal{S}$-*monoclus* and the extended nucleolus is that the latter lexicographically maximizes the ordered vector of excesses associated with each subgame separately, while the former does the maximization at once for all subgames. However, both procedures yield the same outcome.

Proposition 2.7: Given any (N, v), $\mathcal{S}$-*monoclus* ($\mathcal{M}^{\mathcal{S}}(\cdot)$) is the extended nucleolus ($\mathcal{M}(\cdot)$).

Proof: Suppose for some $\mathcal{M}^{\mathcal{S}}(T) \neq \mathcal{M}(T)$ for some $T \subseteq N$. Consider the allocation scheme: $\mathcal{X}(T') = \mathcal{M}^{\mathcal{S}}(T')$ for all $T' \neq T$ and $\mathcal{X}(T) = \mu(T)$ where $\mu(T)$ is the nucleolus of subgame

(T, v). Note that for any $T' \neq T$ and $S \subseteq T'$, $e(\mathcal{X}(T'); S) = e\big(\mathcal{M}^{\mathcal{S}}(T'); S\big)$. Then, by definition of the nucleolus we have $\varepsilon(\mathcal{X}) \succ_L \varepsilon\big(\mathcal{M}^{\mathcal{S}}\big)$. $\square$

At a first glance, defining $\mathcal{S}$-*monoclus* may seem artificial and unnecessary. However, its construction, and its equivalence to the extended nucleolus are important to understand the general nucleolus we define in the next subsection, and to compare it with $\mathcal{M}$.

2.4.2 The $\mathcal{K}$-monoclus: an APAS whenever a game admits one

By Proposition 2.3, we say an allocation scheme is absence-proof if and only if we have $g(P, Q \setminus P) - \sum_{i \in L}\big(\mathcal{X}_i(Q) - \mathcal{X}_i(P)\big) \geq 0$ for all $L \subseteq P \subset Q \subseteq N$. We can rewrite the inequality as follows:

$$v(Q) - v(P) - v(Q \setminus P) - \textstyle\sum_{i \in L} \mathcal{X}_i(Q) + \sum_{i \in L} \mathcal{X}_i(P) \geq 0$$

$$\Leftrightarrow \textstyle\sum_{i \in Q \setminus P} \mathcal{X}_i(Q) + \sum_{i \in P \setminus L} \mathcal{X}_i(Q) - \sum_{i \in P \setminus L} \mathcal{X}_i(P) - v(Q \setminus P) \geq 0$$

$$\Leftrightarrow e(\mathcal{X}(Q); Q \setminus P) + \textstyle\sum_{i \in P \setminus L} \mathcal{X}_i(Q) - \sum_{i \in P \setminus L} \mathcal{X}_i(P) \geq 0$$

Renaming $Q = T$, $Q \setminus P = S$, and $P \setminus L = K$, we have $\mathcal{X}(\cdot)$ is absence proof if and only if $e(\mathcal{X}(T); S) + e(\mathcal{X}(T); K) - e(\mathcal{X}(T \setminus S); K) \geq 0$ for all $S \subset T \subseteq N$, $(T \setminus S) \neq K \in 2^{T \setminus S}$. Now, let $\mathcal{K} \stackrel{\text{def}}{=} \big\{(K, S, T): S \subset T \subseteq N, (T \setminus S) \neq K \in 2^{T \setminus S}\big\}$. Then, the Σ-monotonicities of $\mathcal{X}$ corresponding to elements in $\mathcal{K}$ are defined as follows:

$$\Sigma mon\big(\mathcal{X}, (K, S, T)\big) = e(\mathcal{X}(T); S) + e(\mathcal{X}(T); K) - e(\mathcal{X}(T \setminus S); K) \tag{7}$$

Indeed, the manipulation argument is clear when a Σ-monotonicity is negative. If S leaves the game T and produces $v(S)$ outside the allocation procedure, their loss is $e(\mathcal{X}(T); S)$. Also, the loss of K due to the absence of S is $e(\mathcal{X}(T); K) - e(\mathcal{X}(T \setminus S); K)$. If the total loss of agents in $S \cup K$ is negative, i.e., $\Sigma mon\big(\mathcal{X}, (K, S, T)\big) < 0$, then they would manipulate by absence of S at T.

Let $\kappa(\mathcal{X}) \in \mathbb{R}^{|\mathcal{K}|}$ be the vector that has all the Σ-monotonicities of $\mathcal{X}$ corresponding to elements in $\mathcal{K}$ as its coordinates in a weakly increasing order. Then, $\mathcal{K}$-*monoclus* is defined as follows:

$$\mathcal{M}^{\mathcal{K}}(N, v) = \big\{\mathcal{X} \in \mathbb{X}\big(I(N, v)\big): \kappa(\mathcal{X}) \succcurlyeq_L \kappa(Y) \text{ for all } Y \in \mathbb{X}(I(N, v))\big\} \tag{8}$$

Note that the set $\kappa \stackrel{\text{def}}{=} \big\{\kappa(\mathcal{X}) \in \mathbb{R}^{|\mathcal{K}|}: \mathcal{X} \in \mathbb{X}\big(I(N, v)\big)\big\}$ is convex and compact. Then, the following proposition follows immediately by construction of the $\mathcal{K}$-monoclus.

Proposition 2.8: $\mathcal{M}^{\mathcal{K}}(N, v)$ exist and unique at each game with a nonempty imputation set $I(N, v)$. Moreover, if a game (N, v) admits an APAS, then $\mathcal{M}^{\mathcal{K}}(N, v)$ is an APAS.

2.4.3 Comparing the $\mathcal{K}$-monoclus and the extended nucleolus

Note that $\Sigma mon\big(\mathcal{X}, (\emptyset, S, T)\big) = \Sigma mon\big(\mathcal{X}, (S, T)\big)$. Hence, for any allocation scheme $\mathcal{X}$, each component of the vector $\varepsilon(\mathcal{X})$ is also a component of $\kappa(\mathcal{X})$. However, $\kappa(\mathcal{X})$ has extra components corresponding to the cases $K \neq \emptyset$. When the size of the society grows, the number of these extra components in $\kappa(\mathcal{X})$ grows rapidly, and that makes $\mathcal{K}$-monoclus much harder to calculate compared to the extended nucleolus in general. However, if the number of agents is sufficiently small and/or the game is highly symmetric across partitions of the agents, calculation is feasible. Note that for any 2-person game $\varepsilon(\mathcal{X}) = \kappa(\mathcal{X})$, and hence $\mathcal{M}^{\mathcal{K}} = \mathcal{M}$.

Here, we will analyze the differences and similarities between these two solutions in two sets of games. In the 3-person bilateral assignment games they always differ. In a certain subset of pessimistic bankruptcy games (see Example 2.4) both solutions coincide.

Example 2.3: Let $\bar{\mathcal{G}}$ be the set of 3-person bilateral games as in Section 2.3. For any $(N, v) \in \bar{\mathcal{G}}$, wlog $v(N) = v(1,2) = u_2$, $v(1,3) = u_1$, and $v(S) = 0$ otherwise, with $u_2 \geq u_1 > 0$.

The extended nucleolus of the game above is simple to calculate and is as follows:

$$\mathcal{M}(N) = \big((u_2 + u_1)/2, (u_2 - u_1)/2, 0\big), \mathcal{M}(12) = (u_2/2, u_2/2, *), \mathcal{M}(13) = (u_1/2, *, u_1/2).$$

Note that agent 1's payoff at game N, and $\{1,2\}$ are never the same. Recall from Section 2.3 that those payoffs should be the same at an absence-proof allocation scheme. Hence, $\mathcal{M}^{\mathcal{K}}$ and $\mathcal{M}$ never coincide on $\bar{\mathcal{G}}$. Indeed, by Proposition 2.4, $\mathcal{M}^{\mathcal{K}}$ can be described by 4 numbers; $\mathcal{M}_1^{\mathcal{K}}(N) = \mathcal{M}_1^{\mathcal{K}}(12) = X_1$, $\mathcal{M}_2^{\mathcal{K}}(N) = \mathcal{M}_2^{\mathcal{K}}(12) = X_2$, $\mathcal{M}_1^{\mathcal{K}}(13) = x_1$, and $\mathcal{M}_3^{\mathcal{K}}(13) = x_3$. The $\mathcal{K}$-monoclus is as follows:

$$X_1 = (2u_2 + u_1)/4, X_2 = (2u_2 - u_1)/4, x_1 = x_3 = u_1/2 \quad \text{if } u_2 \geq 5u_1/2.$$

$$X_1 = (u_2 + 2u_1)/3, X_2 = 2(u_2 - u_1)/3, x_1 = (4u_1 - u_2)/3, x_3 = (u_2 - u_1)/3 \quad \text{otherwise.}$$

Excluding the (K, S, T)s with trivially zero Σ-monotonicities, it is an easy exercise to check that the minimum value for the Σ-monotonicities is attained at the triples (1,0,13)and (3,0,13) in the first case; (3,0,13), (13,0,123), and (2,3,123) in the second case.

Example 2.4: *(Aumann, and Maschler 1985).* A bankruptcy problem is defined by an estate of size E to be divided among the claimants, and d_i denotes the claim of individual i. The associated pessimistic bankruptcy game is $v(S) = max\{E - \sum_{i \in N \setminus S} d_i, 0\}$, for all $S \subseteq N$. Let $\tilde{\mathcal{G}}$ be the set of all such games where $d_i \geq 2E/|N|$.

We first define the set of coalitions with minimal excess of the nucleolus:

$$\mathcal{D}(\mu(N, v)) = argmin_{S \in 2^N \setminus \{N, \emptyset\}} e(\mu(N); S)$$

For any $(N, v) \in \tilde{\mathcal{G}}$, we know that $\mathcal{D}\big(\mu(N, v)\big) = \big\{\{i\}: i \in N\big\}$, and the nucleolus is $\mu_i(N, v) = v(N)/|N|$ for all $i \in N$ (Arin and Iñarra 1998).

Proposition 2.9: For any $(N, v) \in \tilde{\mathcal{G}}$, $\mathcal{M}_i(S) = \mathcal{M}_i^{\mathcal{K}}(S) = v(S)/|S|$ for all $S \subseteq N$, and for all $i \in N$.

Proof: Proposition trivially holds for the case $|N| = 1$ or $|N| = 2$. Take any $(N, v) \in \tilde{\mathcal{G}}$ with $|N| \geq 3$. We first show that $(S, v) \in \tilde{\mathcal{G}}$ for all $S \subseteq N$. By construction of $v(\cdot)$, for any $S \subseteq N$, the subgame (S, v) is a pessimistic bankruptcy game associated with $E' = v(S)$ and the vector

$\{d_i\}_{i\in S}$. Consider the subgame $N\setminus j$ for some $j\in N$. We know that $v(N\setminus j)=E'=E-d_j$. Note that $E'\leq E-\frac{2E}{|N|}$, then $\frac{2E'}{|N|-1}\leq\frac{2(|N|-2)E}{(|N|-1)|N|}<\frac{2E}{|N|}\leq d_i$ for all $i\in(N\setminus j)$. Hence, $(N\setminus j,v)\in\tilde{\mathcal{G}}$. Now, take any $S\subseteq N$. Remove one agent at a time from N until we finally reach S. At each step, the subgame we have is in $\tilde{\mathcal{G}}$. Therefore, $(S,v)\in\tilde{\mathcal{G}}$, and hence, $\mathcal{M}_i(S)=v(S)/|S|$. Also, it is apparent from the above inequality that for any $S\subset T$ we have $v(T)/|T|\geq v(S)/|S|$, and also if $v(T)>0$, $\mathcal{M}_i(T)>\mathcal{M}_i(S)$ for all $i\in S$. Hence, $\mathcal{M}$ is PM.

Claim: Take any $(K,S,T)\in\mathcal{K}$. We have $\Sigma mon\big(\mathcal{M},(K,S,T)\big)\geq\Sigma mon\big(\mathcal{M},(\emptyset,i,T)\big)=e(\mathcal{X}(T);i)=v(T)/|T|$.

Proof of Claim: Let $(K,S,T)\in\mathcal{K}$. We know that $d_i\geq 2E/|N|$ for all $i\in T$. Then, $v(j)=0$ as $E-\sum_{i\in N\setminus j}d_i\leq 0$ for all $j\in N$. As $(T,v)\in\tilde{\mathcal{G}}$, and $\mathcal{D}_1\big(\mu(T,v)\big)=\big\{\{i\}:i\in T\big\}$ we have $\Sigma mon\big(\mathcal{M},(\emptyset,i,T)\big)=e(\mathcal{M}(T);i)=v(T)/|T|\leq e(\mathcal{M}(T);S)$ for all $i\in T$, $S\subset T$. Also, as $\mathcal{M}$ is PM we have $e(\mathcal{M}(T);K)-e(\mathcal{M}(T\setminus S);K)\geq 0$.

Let $\{\mathbb{T}_0,\mathbb{T}_1,\dots,\mathbb{T}_m\}$ be a partition of 2^N s.t. for any l, and $T,T'\in\mathbb{T}_l$ we have $v(T')/|T'|=v(T)/|T|$, and for any $T\in\mathbb{T}_l$, $T'\in\mathbb{T}_{l+1}$ we have $v(T')/|T'|>v(T)/|T|$. Note that $\mathbb{T}_0=\{T\subseteq N:v(T)=0\}$ is non-empty as $\{i\}\in\mathbb{T}_0$ for all $i\in N$, and $\mathbb{T}_m=\{N\}$. Also, for every $(K,S,T)\in\mathcal{K}$ with $T\in\mathbb{T}_0$ we have $\Sigma mon\big(\mathcal{M},(K,S,T)\big)=0$.

By Claim 1, the minimum value for $\Sigma mon\big(\mathcal{M},(K,S,T)\big)$ with $T\notin\mathbb{T}_0$ is attained at $(\emptyset,i,T)\in\mathcal{K}$ with $T\in\mathbb{T}_1$. Suppose for a contradiction $\mathcal{M}(T)\neq\mathcal{M}^{\mathcal{K}}(T)$ for some $T\in\mathbb{T}_1$. Then, $e\big(\mathcal{M}^{\mathcal{K}}(T);i\big)<v(T)/|T|$ and $\kappa(\mathcal{M})\succ_L\kappa\big(\mathcal{M}^{\mathcal{K}}\big)$. Again by the Claim, the minimum value for $\Sigma mon\big(\mathcal{M},(K,S,T)\big)$ with $T\notin(\mathbb{T}_0\cup\mathbb{T}_1)$ is attained at $(\emptyset,i,T)\in\mathcal{K}$ with $T\in\mathbb{T}_2$. Suppose for a contradiction $\mathcal{M}(T)\neq\mathcal{M}^{\mathcal{K}}(T)$ for some $T\in\mathbb{T}_2$. Then, $e\big(\mathcal{M}^{\mathcal{K}}(T);i\big)<v(T)/|T|$ and $\kappa(\mathcal{M})\succ_L\kappa\big(\mathcal{M}^{\mathcal{K}}\big)$. The argument applies recursively, hence we are done. □

3 Exchange Economies with Private Endowments

Basic notions In order to get rid of notational complexity, we prefer to define the basic concepts and most of the models in words. An exchange economy is formed of a set of individuals N, individualized endowments, a feasible consumption space, and preferences of the individuals on this consumption space. An allocation is a redistribution of the total endowments, possibly with some restrictions on individual consumptions, and a vector of balanced monetary transfers (if available in the model). An allocation rule assigns an allocation or a set of allocations to all problems in a specified domain.

Pareto optimality An allocation is Pareto optimal (PO) if there is no other feasible allocation that makes an agent strictly better-off without hurting some other agent.

Core stability An allocation is in the core if for any group of agents there is no way to redistribute their total endowment among the group in a way Pareto superior to their allocation.

Böhm-Bawerk's horse market The traded goods are indivisible identical objects (horses here). Society is formed of potential sellers and buyers. Sellers own a horse each while the buyers own none. Each agent wants to consume at most one horse. Preferences are represented by a number corresponding to reservation price for the sellers, and willingness to pay for the buyers. Monetary transfers are allowed, and preferences are quasilinear in money. An allocation is a redistribution of the horses such that each agent has at most one horse, and a vector of transfers that adds up to 0.

House assignment problem *(Shapley and Shubik 1971)* The traded goods are indivisible identical objects (houses here). Each agent owns exactly one house. For each agent, preferences are represented by n numbers corresponding to their willingness to pay for each house. Monetary transfers are allowed, and preferences are quasilinear in money. An allocation is a redistribution of the houses such that each agent gets exactly one house, and a vector of transfers that adds up to 0.

Housing markets *(Shapley and Scarf 1974)* The traded goods are indivisible identical objects (houses here). Each agent owns exactly one house, and his ordinal preference is a linear order over the set of all houses. Monetary transfers are not allowed, and an allocation is a redistribution of the houses among the agents.

Classical exchange economies: An economy is a triple $\varepsilon = (N, e, R)$. $N \in \mathcal{N}$ denotes the set of individuals, where $\mathcal{N}$ is the set of all finite subsets of $\mathbb{Z}$. $e = \{e_i\}_{i\in N}$ is the profile of private endowments, where $e_i \in \mathbb{R}^{\ell}_{+}$ for all i, and e^S denote the endowment profile restricted to $S \subseteq N$. Each individual has a complete and transitive preference relation R_i on $\mathbb{R}^{\ell}_{+}$, and P_i denotes the strict counterpart of R_i. Let $\mathcal{R}$ denote the set of admissible preferences for each individual. Given a society $N \in \mathcal{N}$, a preference profile is a vector $R \in \mathcal{R}^N$, and R^S is the profile restricted to $S \subseteq N$. We denote the restriction of the economy $\varepsilon = (N, e, R)$ to $S \subseteq N$ by ε^S i.e., $\varepsilon^S = (S, e^S, R^S)$. Given an economy $\varepsilon = (N, e, R)$, an allocation $x \in \mathbb{R}^{n\ell}_{+}$ is a vector s.t. $x_i \in \mathbb{R}^{\ell}_{+}$ for all $i \in N$, and $\sum_{i\in N} x_i = \sum_{i\in N} e_i$. An allocation rule $\varphi(\cdot)$ assigns an allocation to each economy ε.[8]

3.1 On the core, the competitive equilibrium, and AP

The competitive equilibrium is without dispute the most fundamental solution concept in exchange economies. In most cases it exists, and ensures Pareto optimality. Moreover, for all the problems we will discuss here, it is well-known that the competitive allocations are always core stable. However, it is not immune to manipulations in all respects. In classical exchange economies, an agent can manipulate the competitive equilibrium by withholding or destroying his endowment as discussed in Postlewaite (1979). There, Postlewaite also discusses a group manipulation strategy. A group of agents may perform a trade prior to coming to the market. With their "new" endowments, they can be better off at the allocation the rule assigns. Agents

[8] We prefer defining rules as functions instead of correspondences to keep things simple.

can also manipulate the competitive equilibrium by misrepresenting their preferences (Hurwicz 1972).

In a Böhm-Bawerk market, competitive allocations are determined by a set of prices which equalizes the number of sellers and buyers that are willing to trade. Here, the set of competitive allocations and the core stable allocations coincide. Manipulation by withholding and destroying has no bite as each seller owns a single indivisible unit. An active coalition that performs pre-trade consists of a subset of both sellers and buyers. This trade will make either the sellers or the buyers unhappy, or everyone remains equally happy as there is a uniform market price. Also, strategy-proof mechanisms that are immune to manipulation by misrepresenting preferences exist (see e.g. Moulin (1995)).

In this setting, Shapley and Shubik (1971) discuss some weakness of the core. They say, "The core is based on what a coalition can do, not what it can prevent". Here is a summary of their argument: There are four suppliers and four buyers. Reservation prices for suppliers in an increasing order, and willingness to pay for buyers in a decreasing order are as follows: 2, 3, 4, 7 for suppliers; 12, 9, 8, 3 for buyers. Here, the competitive price ranges from 4 to 7. If the seller with the reservation price of 4 was not involved, the competitive price would range from 7 to 8. Instead of using his bargaining power in the market directly, by not appearing in the scene, he would help increase the bargaining power of the remaining sellers. Below, we take their example to an extreme case.

Example 3.1: (*Horse market*) There are three potential suppliers with a reservation price of 0, and two potential buyers with a willingness to pay of 1.

The unique competitive price is 0. The buyers get a horse each and pay nothing. Hence, in the unique core allocation, buyers equally share the entire surplus of 2 units. If two of the suppliers stay outside of the market with their horses, the remaining supplier sells his horse at a price of 1 at the unique competitive allocation. As AP implies core stability by the very general definition, there is no absence-proof allocation rule here.

Indeed, the problem in this example induces exactly the same TU cooperative game given in Example 2.1; suppliers correspond to workers, and buyers correspond to firms. We argued in Section 2.1 that the formulation of manipulation in TU games by the inequality (1) cannot be directly applied here. However, as the valuation for the good is exactly the same for each buyer, after the allocation process, the manipulating coalition cannot create extra surplus by passing a horse from one buyer to another. Hence, the argument that rules out AP in both examples is equivalent. The following example reflects exactly the same idea in house assignment problems.

Example 3.2: (*House assignment*) There are five agents, each endowed with a house. Society is partitioned in two distinct sets, say with three agents in set W and two agents in set F. An agent from W is willing to pay 1 unit for a house owned by an agent from F, and 0 for a house owned by an agent from W. And vice versa for an agent in F.

Obviously, the problem above induces the same TU game with the one in Example 2.1, and all the "reduced" problems induce the same subgames. Again for this specific problem, formulation of AP is the same as in inequality (1).

Example 3.3: (*Auction*) There is a single seller who owns a single indivisible object. Assume for simplicity his valuation for the good is 0. There are $n \geq 3$ buyers in the market, and their willingness to pay is as follows: $b_1 > b_2 > \cdots > b_n$.

This is a special case of Böhm-Bawerk's horse market. In a core allocation buyer 1 gets the good paying the seller at least b_2, and other buyers pay nothing. Here, there are a number of possibilities for manipulation. We consider just the following trivial case: All the buyers except n stays out. Agent n gets the good by paying at most b_n units. After the trade, he passes the good to buyer 1, and buyer 1 pays the others, say $b_3/(n-1)$ units. This move makes all the buyers strictly better-off.

There is a similar source of collusion in the auction theory. A manipulating coalition is called a "ring", and members of a ring never bids against each other. If one of the members get the object, they would perform an (or a series of) unofficial auction(s) afterwards, and make all the members better-off. In the example above, if set of all the buyers forms a ring, none of them would bid against agent n, resulting in a similar outcome as we suggested.

Proposition 3.1: There is no absence-proof allocation rule in Böhm-Bawerk's horse market, in a single seller auction with a single object, and in house assignment problems.

Postlewatie (1979) showed that in a classical exchange economy, no allocation rule satisfies withholding-proofness along with Pareto optimality and individual rationality. Note that any core allocation is Pareto optimal and individually rational. In both problems above, AP resembles withholding-proofness while the withholding entity is a group rather than an individual. Moreover, in Example 3.1, and 3.2 as the manipulating coalition does not utilize the goods outside, specific to these examples, the manipulation argument resembles destruction of endowments. As AP implies core stability, the impossibility result above is not surprising.

Group manipulations, as well as the total secession in the core give rise to some transaction cost related to the means of agreement. This cost grows with the size of the manipulating coalition. In both examples the manipulating coalition consists of only three agents. This number is big compared to the size of the society. However, even if the society is formed of 199 agents, with a 100 of one type and 99 of the other, a coalition of three agents would still manipulate with the same argument.

3.1.1 Housing markets

Here, there is a unique core allocation if individual preferences are strict. This allocation can be implemented by the famous *top trading cycle* (TTC) algorithm introduced by Shapley and Scarf (1974). The TTC algorithm is as follows: Let every agent in N be represented by a node in a directed graph. From each agent i, draw a directed link to the agent j who owns the top house in i's linear order. This will create at least one cycle. Within each cycle perform the trade so that each agent gets his top house. All agents who get their top choice in the first round leaves the scene with their new house (unless i's top choice is his own house, in that case he leaves with his own house). Delete the houses that left the scene from the preference of the agents who were not in a cycle in the first round. Now, apply the same procedure among them. This algorithm stops in a finite number of rounds, and returns the unique core allocation.

The direct revelation mechanism through TTC algorithm (henceforth *core mechanism*) is also known to be group strategy-proof; that no coalition can gain by jointly misrepresenting their preferences. However, a group of agents can manipulate the core mechanism by performing a trade prior to the implementation of the mechanism (see Moulin (1995)). This move never makes every agent in the coalition strictly better-off.

Example 3.4: Consider the following economy where h_i represents the house that agent i owns, and P_i represent agent i's preferences.

$\boldsymbol{P_1}$	$\boldsymbol{P_2}$	$\boldsymbol{P_3}$
$\boldsymbol{h_2}$	h_1	h_2
$\boldsymbol{h_3}$	.	h_1
$\boldsymbol{h_1}$	.	h_3

In the core mechanism, agents 1 and 2 trade houses getting their top choice and agent 3 is left out with his own house. If the coalition $\{2,3\}$ agrees on agent 2 to stay out, agent 1 gets h_3 and agent 3 gets h_1 in the core mechanism. Afterwards, agent 3 gives h_1 to agent 2 in return for h_2. This move results in a (weak) Pareto improvement for the coalition $\{2,3\}$. Note that the final outcome after the manipulation can also be achieved by a pre-trade between agents 2 and 3 prior to the implementation of the mechanism.

Absence-proofness (weak vs. strong) In the absence of monetary transfers, a weak Pareto improvement (that not all the agents from the manipulating coalition strictly benefits) is not always considered as a true motivation for a group manipulation. One particular reason is that agents have preferences over the houses but not on the allocation. Here, we say a rule is *strongly absence*-proof if it is immune to manipulations by weakly Pareto improving moves. If manipulation requires a strict Pareto improvement, the corresponding stability concept is *weak absence-proofness*.

Proposition 3.2: There is no strongly absence-proof allocation rule in housing markets, while core mechanism yields the unique weakly absence-proof allocation rule.

Proof: As AP induces core stability, and there is a unique core allocation at each problem, Example 3.4 proves the first statement. Let N_1 denote the set of agents that gets his top house at the first round of the TTC mechanism, N_2 denote those who get the "restricted" top choice in the second round and so on. Suppose a coalition S is able to make a strict Pareto improvement by leaving a subgroup T out. Then, $S \cap N_1 = \emptyset$. As $T \cap N_1 = \emptyset$, agents who get their top choice at problem $N \setminus T$ in the core mechanism is again N_1. Then, no agent from N_2 can get a better house in the absence of T. Given, $S \cap N_2 = \emptyset$, by a similar argument we have $S \cap N_3 = \emptyset$, and so on. Uniqueness follows from the fact that there is a unique core allocation. □

3.1.2 Classical exchange economies

As we discussed earlier, the competitive equilibrium is proven to be vulnerable to individual and group manipulations in many ways in this specific model. Here, the set of competitive equilibrium allocations (when it exists) is a strict subset of the set of core allocations. However, when the economy is large enough (when the effect of a single agent on the competitive price is negligible) the set of competitive allocations converges to the set of core allocations. As absence-proofness implies core, our intuition tells that a selection from competitive allocations (henceforth Walrasian allocation) is our only hope for an absence-proof allocation rule.

Definition 3.1: An allocation rule $\varphi(\cdot)$ is AP on a domain of preferences $\mathcal{R}$ if for any economy $\varepsilon = (N, e, R)$ with $R \in \mathcal{R}^N$, for any $T \subseteq N$, and $K \subseteq N \setminus T$, there is no $y \in \mathbb{R}_+^{(k+t)\ell}$ with $\sum_{i\in(K\cup T)} y_i = \sum_{i\in K} \varphi_i(\varepsilon^{N\setminus T}) + \sum_{i\in T} e_i$ s.t. y Pareto dominates $\{\varphi_i(\varepsilon)\}_{i\in(K\cup T)}$ for agents in $K \cup T$.

Remark 3.1: AP implies PO and core stability by definition. Just set $T = N$, $K = \emptyset$ for PO, and for all T set $K = \emptyset$ for core stability.

The following example illustrates that the divisibility of the goods enlarges the manipulation options. In an economy with just three individuals and even where all agents have the same fine Cobb-Douglas preferences, the Walrasian allocation is manipulable.

Example 3.5: $\ell = 2, N = 3, \{e_1, e_2, e_3\} = \{(10,10), (35,5), (15,15)\}$. $u_i = x_i y_i$ for all i.

Check that the Walrasian allocation $\phi(\cdot)$ and the induced utilities for the problem N and $S = \{1,2\}$, with the prices $p(N) = (1,2)$ and $p(S) = (1,3)$ as follows:

$\boldsymbol{S}$	$\boldsymbol{\phi_1(S)}$	$\boldsymbol{\phi_2(S)}$	$\boldsymbol{\phi_3(S)}$	$\boldsymbol{u_1(S)}$	$\boldsymbol{u_2(S)}$	$\boldsymbol{u_3(S)}$
$\{\mathbf{1,2,3}\}$	(15,7.5)	(22.5,11.25)	(22.5,11.25)	112.5	253.125	253.125
$\{\mathbf{1,2}\}$	$(20, 20/3)$	$(25, 25/3)$	$*$	400/3	625/3	$*$

Table 3.1

Note that $\phi_1(\{1,2\}) + e_3 = (35, 21.\overline{6})$. Consider the following redistribution of this total to individuals 1, and 3: $z_1 = (15, 7.\overline{6})$, and $z_3 = (20,14)$. Hence, the Walrasian allocation is manipulable by agents 1 and 3 as we have, $u_1(z_1) > u_1(N)$ and $u_3(z_3) > u_3(N)$.

Example 3.6: There are 11 agents each of whom is endowed with 1 kg of beans and 1 kg of rice. All of the agents have linear preferences as follows: $u_1 = r_1 + 10b_1$, $u_i = 10r_i + b_i$ for $i = 1, \dots, 11$.

The unique competitive price in the above economy is $(1, p) = (1,10)$, where p is the price of rice. At this price, agent 1 exploits the entire surplus in the market, and the remaining agents are left with their initial utility level of 11. Let $S = \{2, \dots, 11\}$, and only $K \subset S$ comes to the market while agent 1 is always active in the market. Then, the unique competitive price is $p = |K|$. Hence, all the agents in K has a positive profit from the trade. This profit can be redistributed in a way that makes every agent strictly better-off with respect to their initial utility level as the utility is linear.

Proposition 3.3: On the domain of linear preferences and the Cobb-Douglas preferences, the Walrasian allocation rule is not AP.

Thomson (2013) defines the same manipulating argument, and the corresponding stability property withdrawal-proofness, where the manipulating party consists of only two agents. Note that withdrawal-proofness is weaker than AP. He proves that the Walrasian allocation is not withdrawal-proof on the domain of homothetic preference. So, our results coincide with Thomson (2013).

As almost all other stability arguments, the manipulation argument relies on the perfect knowledge of agents about other agents' characteristics (endowments and preferences) and the allocation method. Example 3.6 illustrates an important characteristic of manipulation by absence. Note that, there, any subset S' of S can manipulate by leaving any strict subset $T \subset S'$ out of the allocation process. This suggests that, in some instances, agent's rough idea about the "type" of other agents in the market would be a sufficient motivation to take a manipulating action.

Sertel and Yıldız (1999) discuss the welfare effect of an additional agent that brings new trading opportunities on the existing agents. If the allocation rule always assigns core allocations, we expect some agents to benefit from the appearance of a newcomer (unless in the degenerate case where no existing agent is affected at all). They show that existing agents who have "sufficiently similar types" with the entrant are hurt, and the others benefit as their trading opportunities expands. Their discussion relies on the fact that the population-monotonicity is too demanding in this setting. Hence, we cannot expect all the existing agents to benefit regardless of the entrants type. This fact is quite transparent in Example 3.6. Suppose only agents 2 and 3 are in the market initially. As they are exactly of the same type no trade occurs. If agent 1 arrives in the market, both existing agents would benefit. However, if we add 8 more agents (agents 4 to 11), this would hurt agent 2 and 3, while agent 1 benefits from their arrival.

Just like in TU games, PO and PM implies core stability in this context (see Proposition 3.4 below). However, as we discuss in Example 3.7 below, the logical relation between AP and PM breaks down here. Hence, even the very demanding property PM does not guarantee avoiding manipulation by absence.

Definition 3.2: An allocation rule $\varphi(\cdot)$ is PM on a domain of preferences $\mathcal{R}$ if for any economy $\varepsilon = (N, e, R)$ with $R \in \mathcal{R}^N$, for any $S \subseteq N$ and $i \in S$, we have $\varphi_i(\varepsilon) R_i \varphi_i(\varepsilon^S)$.

Proposition 3.4: If $\varphi(.)$ is PM and PO on a domain of preferences $\mathcal{R}$, $\varphi(.)$ is a core selection on $\mathcal{R}$.

Proof: Let $\varphi(\cdot)$ be PM and PO on $\mathcal{R}$. Take any $\varepsilon = (N, e, R)$ with $R \in \mathcal{R}^N$, let $S \subseteq N$, and x be an allocation in ε^S. By PO we have, $\varphi(\varepsilon^S)$ is not Pareto dominated by x. As by PM $\varphi_i(\varepsilon) R_i \varphi_i(\varepsilon^S)$ for all $i \in S$, $\{\varphi_i(\varepsilon)\}_{i \in S}$ is not Pareto dominated by x either. □

Example 3.7: $\ell = 2$, $N = 3$, $\{e_1, e_2, e_3\} = \{(10,10), (35,5), (15,15)\}$, $u_i = x_i y_i$ for all i. Consider the following allocation scheme and the induced utilities:

S	$\varphi_1(S)$	$\varphi_2(S)$	$\varphi_3(S)$	$u_1(S)$	$u_2(S)$	$u_3(S)$
$\{1,2,3\}$	$(10\sqrt{2}, 5\sqrt{2})$	$(60 - 25\sqrt{2}, 30 - 25\sqrt{2}/2)$	$(15\sqrt{2}, 15\sqrt{2}/2)$	100	303.7	225
$\{1,3\}$	$(10\sqrt{3}, 10\sqrt{3}/3)$	$(45 - 10\sqrt{3}, 15 - 10\sqrt{3}/3)$	$*$	100	255.4	$*$
$\{1,3\}$	$(10,10)$	$*$	$(15,15)$	100	$*$	225
$\{2,3\}$	$*$	$(50 - 15\sqrt{5/2}, 20 - 6\sqrt{5/2})$	$(15\sqrt{5/2}, 6\sqrt{5/2})$	$*$	276.3	225
$\{1\}$	$(10,10)$	$*$	$*$	100	$*$	$*$
$\{2\}$	$*$	$(35,5)$	$*$	$*$	175	$*$
$\{3\}$	$*$	$*$	$(15,15)$	$*$	$*$	225

Table 3.2

Note that φ is PM, PO and $\varphi_1(\{1,2\}) + e_3 \geq (32.3,20.7)$. Consider the redistribution of this total to individuals 1 and 3 as $z_1 = (14.3,7.7)$ and $z_3 = (18,13)$. We have $u_1(z_1) > u_1(N)$, $u_3(z_3) > u_3(N)$, and hence φ is not AP.

4 Fair Division Problems

Basic Notions A set of individuals $N \in \mathcal{N}$ have equal claims on a fixed supply of commonly owned goods $\Omega \in \mathcal{C}$, where $\mathcal{C}$ stands for the consumption space, and $\mathcal{N}$ denotes the set of all finite subsets of $\mathbb{Z}^+$. For now, we do not impose a structure on $\mathcal{C}$ so that we keep our main result as general as possible. However, we want the reader to keep in mind that we are dealing with the allocation of a bundle of perfectly divisible goods in $\mathbb{R}_+^\ell$, or a finite set of indivisible objects, or a combination of both. Two important cases we do not include in Theorem 4.1 is the fair division of a heterogeneous, and divisible commodity (generally known as cake cutting, or land division problem), and the case where we do not necessarily allocate Ω wholly. Indeed, with some further specifications the result will still hold. However, we want to keep the idea (and the notation) as simple as possible. A very critical assumption for the manipulation argument to be conceptually proper is that consumption is private, and once an agent receives the good, he has the complete right to consume or transfer it to another agent. So, we are dealing with excludable and rival goods. Moreover, we can safely introduce monetary transfers into the model, and say along with the goods, a total amount $M \in \mathbb{R}$ of money is distributed. In that case M is embedded as a separate component in $\mathcal{C}$. Each individual i in the maximal society has a complete and transitive preference relation R_i on $\mathcal{C}$, and P_i denotes the strict counterpart of R_i. $\mathcal{R}$ denotes the set of admissible preferences for each individual. Given a society $N \in \mathcal{N}$, a preference profile is a vector $R \in \mathcal{R}^N$, and R^S is the profile restricted to $S \subseteq N$. Then, a fair division model is $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, and a specific problem is simply a triple (N, Ω, R). We also assume that preferences are strictly increasing in money in case transfers are allowed.

For a fixed (N, Ω, R), an *allocation* $x = (x_i)_{i \in N}$ is a vector s.t. the sum and/or union of the total allocated goods is Ω. Specifically, in the context of indivisible good, no two agents share the same good while some agents may receive no good at all. Also, no agent receives a negative amount of any divisible good. $X(N, \Omega, R)$ denotes the *set of allocations*. For each $x \in X(S, \Omega, R^S)$, and $K \subseteq S \subseteq N$, $\sum_K x\,(S)$ denotes the sum and/or union of goods (and possibly money) that the coalition K gets in allocation x at the "reduced" problem (S, Ω, R^S) where agents in S are the only claimants. An allocation $x \in X(N, \Omega, R)$ is *Pareto optimal* (*PO*) if x is not Pareto dominated by another allocation i.e., there is no $z \in X(N, \Omega, R)$ such that $z_i R_i\, x_i$ for all $i \in N$ and $z_j P_j x_j$ for some $j \in N$.

Given a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, an *allocation rule* φ is a mapping that assigns a *subset of allocation*s to each problem (N, Ω, R). An allocation rule is Pareto optimal if φ assigns PO allocations to all (N, Ω, R).

In case monetary compensations are not available, for any two consumption bundles x, y in the consumption space, we say that $x > y$ if x is weakly greater in all components, and strictly greater in at least one component. Definition is akin to the vector relation in $\mathbb{R}^{\ell}{}_{+}$, and if we have a set of indivisible objects as a component in x and y, weakly and strictly greater corresponds to the set inclusion relations $\subseteq$ and $\subset$, respectively. Also, $x \gg y$ if x is strictly greater in all components. When compensations are possible, if two bundles have the same transfer, definitions remain the same. If x has strictly more money and x is weakly greater in all other components, we say $x \gg y$.

4.1 On the AP and PM

In an allocation problem with common endowments, absence of a coalition S in the allocation process means that they renounce their claims. Thus, core has no bite here. However, the partial secession of S, meaning only a strict $T \subset S$ is left out, can still be profitable. In that sense, (to my knowledge) absence-proofness is the first core-like stability property in the context of fair division (except withdrawal-proofness in Thomson (2013)).

Definition 4.1: An allocation rule φ is *manipulable* at a problem (N, Ω, R) by a coalition of agents $S \subseteq N$ via absence of $T \subset S$, if there exist $y \in \varphi(N, \Omega, R)$, $y' \in \varphi(N \setminus T, \Omega, R^{N \setminus T})$, and $\{z_i\}_{i \in S} \in X\left(S, \sum_{S \setminus T} y'(N \setminus T)\,, R^S\right)$ (a reallocation of what $S \setminus T$ gets in the allocation y' at problem $\left(N \setminus T, \Omega, R^{N \setminus T}\right)$ to the agents in S) s.t. $z_i R_i\, y_i$ for all $i \in S$, and $z_j P_j y_j$ for some $j \in S$.

Definition 4.2: Given a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, an allocation rule φ is *absence-proof* (*AP*) if it is not manipulable at (N, Ω, R), for any $N \in \mathcal{N}$, $R \in \mathcal{R}^N$ and $\Omega \in \mathcal{C}$.

Note that manipulability is defined here in the weak form; existence of one allocation in the grand game and one allocation in the reduced game is enough. Thus, AP has the strongest possible interpretation. In the examples we provide for our negative results, the allocation rules assign single allocations, although they may assign multiple allocations in general (see CEEI in Section 4.2). Hence, this strong interpretation does not affect the results here. Moreover, it enhances the robustness of AP rules we discuss here in terms of stability.

Proposition 4.1: Given $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, every AP allocation rule is PO.

Proof: Suppose φ is not PO at (N, Ω, R), and say $z \in X(N, \Omega, R)$ Pareto dominates $y \in \varphi(N, \Omega, R)$. Let $S = N$, and fix a $T \subset S$. As $\sum_{N \setminus T} y' = \Omega$, for any $y' \in \varphi\big(N \setminus T, \Omega, R^{N \setminus T}\big)$, we have $z \in X\big(N, \sum_{N \setminus T} y', R^N\big)$. Then, φ is manipulable at (N, Ω, R) by N via absence of T. □

When agents share a fixed supply of goods, it is natural to ask no one to benefit from arrival of additional agents. This is one (strong) interpretation of population-monotonicity as a normative solidarity principle. However, in case monetary compensations are available (and utilities are quasilinear in money), an additional agent who receives a much higher utility from some bundle than all the existing agents may significantly increase the monetary value of the pie to be distributed. Then, it is plausible for an existing agent to benefit from the arrival of the newcomer. In that case, the solidarity principle asks either no existing agent lose, or no one gains. This weaker version appears under different names in the literature; population solidarity, weak population-monotonicity, and even sometimes population-monotonicity (Moulin 1992; Thomson 1995; Tadenuma and Thomson 1993).

Definition 4.3: Given a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, an allocation rule φ is *population-monotonic* (*PM*) if for all for all $\Omega \in \mathcal{C}$; $N, N' \in \mathcal{N}$ with $N \subseteq N'$, $R \in \mathcal{R}^{N'}$, $y \in \varphi(N, \Omega, R^N)$, $y' \in \varphi(N', \Omega, R)$, we have $y_i R_i y'_i$ for all $i \in N$. φ is *weakly population-monotonic* (*wPM*), if we have either $y_i R_i y'_i$ for all $i \in N$, or $y'_i R_i y_i$ for all $i \in N$.

Note that Proposition 4.1 does not hold if we replace AP with PM. Just consider the case $\mathcal{C} = \mathbb{R}^{\ell}_{+}$, and $\varphi_i(N, \Omega, R) = \Omega / n$ for all $i \in N$. φ is obviously PM for $\mathcal{R}$ being the domain of all monotone preferences[9], but not PO for many preference profiles.

Theorem 4.1: Given a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, if a PO allocation rule φ is PM, then it is also AP.

Proof: Fix a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, and let φ be PO and PM. Suppose for a contradiction that φ is not AP. Then, for some (N, Ω, R), $T \subset S \subseteq N$, $y \in \varphi(N, \Omega, R)$, $y' \in \varphi\big(N \setminus T, \Omega, R^{N \setminus T}\big)$, there is $\{z_i\}_{i \in S} \in X\big(S, \sum_{S \setminus T} y'(N \setminus T), R^S\big)$ s.t. $z_i R_i y_i$ for all $i \in S$, and $z_j P_j y_j$ for some $j \in S$. By PM, for all $i \in N \setminus S$, we have $y'_i R_i y_i$. Now, consider the following allocation at problem (N, Ω, R): $x_i = y'_i$ if $i \in N \setminus S$, and $x_i = z_i$ if $i \in S$. Note that $x \in X(N, \Omega, R)$, and it Pareto dominates y. This contradicts that φ is PO. □

Interestingly, although PM works in the opposite direction in TU surplus sharing games and in fair allocation problems, PM implies AP in both problems. In the context of divisible goods ($\mathcal{C} = \mathbb{R}^{\ell}_{+}$), Thomson (2013) introduces *withdrawal-proofness*. While the manipulation idea is the same as in AP, the manipulating coalition consists of only two agents; corresponding to $|S| = 2$, and $|T| = 1$ in Definition 4.1. Note that this property is weaker than AP. He also relates it to PM, but from an opposite direction. He argues that if a coalition S manipulates by withdrawal of agent i, then the agent who stays in should not be worse off by departure of i in

[9] R_i on $\mathbb{R}^{\ell}_{+}$ is said to be monotone if for any $z, z' \in \mathbb{R}^{\ell}_{+}$ with $z < z'$ we have $z' R_i z$, and if $z \ll z'$ we have $z' P_i z$. Moreover, it is strictly monotone if $z < z'$ implies $z' P_i z$.

the restricted problem. Hence, his welfare should be affected in the manner required by PM. This argument is true. However, the critical argument for Theorem 4.1 depends on how the agents in $N \setminus S$ are affected as the resource is fixed.

Theorem 4.1 provides a sufficient condition to block the possibility of manipulation. Finding a simple necessary and sufficient condition for AP is not an easy task in general. However, in the very simple model of allocating a single object where monetary transfers are available, it is possible (see Section 4.3.1).

An easier task is to define a sufficient condition for manipulation. Suppose an additional agent arrives in the allocation process, and an existing agent j receives no less than what he receives before and also more of some divisible good and/or an extra object and/or extra money. Then, all the existing agents except j would be willing to compensate the newcomer out of their total allocation and ask him to stay out.

Proposition 4.2: Given a model $(\mathcal{N}, \mathcal{C}, \mathcal{R})$, let φ be a allocation rule such that for some $N, N' \in \mathcal{N}$ with $N \subseteq N'$, $j \in N$, $\Omega \in \mathcal{C}$, $R \in \mathcal{R}^{N'}$ where R_i is monotone (strictly monotone)[10] for all $i \in N'$. If for some $y \in \varphi(N, \Omega, R^N)$, $y' \in \varphi(N', \Omega, R)$ we have $y_j \ll y'_j$ $(y_j < y'_j)$, φ is not AP.

Proof: Let $T = N' \setminus N$ and $S = N' \setminus \{j\}$. Note that $T \subset S$ and $S \setminus T = N \setminus j$. Then, we have $\sum_S y'(N') = \Omega - y'_j \ll (<) \Omega - y_j = \sum_{S \setminus T} y(N)$. By monotonicity (strict monotonicity) of preferences, φ is manipulable at the problem N' by coalition S via absence of T. ◻

4.2 Perfectly divisible goods with no monetary transfers

Here, we have $\mathcal{C} = \mathbb{R}^{\ell}_{+}$, and for a specific problem (N, Ω, R), an *allocation* $x \in \mathbb{R}^{n\ell}_{+}$ is a vector s.t. $x_i \in \mathbb{R}^{\ell}_{+}$ for all $i \in N$ and $\sum_{i \in N} x_i = \Omega$. Three important solutions for the underlying problem are *competitive equilibrium with equal incomes* (*CEEI*), the *Ω-egalitarian equivalent* (*Ω-EE*) allocation proposed by Pazner and Schmeidler (1978), and the *sequential priority* (*SP*) solution. Given (N, Ω, R), *CEEI* is the set of competitive equilibrium allocations of the economy where each individual is initially endowed with Ω/n. The *Ω-EE* allocation is such that each individual is indifferent between his allocation and $\lambda\,\Omega$ for some number λ. *Ω-EE* picks the highest number λ^* such that a corresponding egalitarian equivalent feasible allocation exists, and assigns one of those allocations among which all the individuals are indifferent. On the domain of strictly monotonic and continuous preferences *Ω-EE* allocation is well-defined and PO. A nice feature of the *Ω-EE* allocation is that on that domain, it is PM (see e.g. Moulin (1995)), while *CEEI* is not (Chichilnisky and Thomson 1987).

Unlike the other two solutions, the *SP* solution is not anonymous. Given a society N, fix a strict priority ordering of the agents in N. Then, at each problem (S, Ω, R) with $S \subseteq N$, just assign Ω to the agent in S who precedes others in the order. Given a maximal society N, and an order on N, the solution is well-defined for each problem at a subsociety $S \subseteq N$, and only at those S. Hence, it resembles the allocation schemes in Section 2. Note that on any domain of

[10] Standard adaptations of these properties in consumption space $\mathbb{R}^{\ell}{}_{+}$ to $\mathcal{C}$. See footnote 8.

preferences this solution is trivially PM. However, it may not be efficient on the domain of monotone preferences. Just consider the case where the first agent in the order is indifferent between any two bundles in $\mathbb{R}_{+}^{\ell}$. Moreover, if the second agent in that order has a strictly monotonic preference, these two agents can manipulate *SP* by the absence of the first agent. Note that on the domain of strictly monotone preferences, the *SP* solution is also PO.

Corollary to Theorem 4.1:

(i) The Ω-*EE* allocation rule is AP on the domain of continuous and strictly monotonic preferences.
(ii) The *SP* solution is AP on the domain of strictly monotonic preferences.

As in exchange economies, competitive idea is vulnerable to manipulation by absence. The following example is used to show that the *CEEI* is not PM on the domain of strictly monotonic preferences by Chichilnisky and Thomson (1987).

Example 4.1: Let $\ell = 2$, $\Omega = (24{,}24)$, $|N| = 4$

$$u_1 = min\{2x + 8, y\}, u_i = min\{18x + 100{,}25y + 132\} \text{ for } i = 2{,}3{,}4.$$

The *CEEI* gives $(2{,}12)$ to agent 1 at game N, and $(1{,}10)$ at game $\{1{,}2{,}3\}$. Note that agent 1 gets less in each good when agent 4 leaves the game.

Corollary to Proposition 4.2: *CEEI* is not AP on the domain of continuous and strictly monotonic preferences.

If everyone has an equal right on the common endowment, it is fair to give them an equal share, but that is not efficient in general. If we assume that individuals seek for welfare improving trade opportunities, they would end up in a competitive equilibrium of the economy where the initial endowment of each individual is Ω/n. Indeed, the *CEEI* is the summary of this process. Although *CEEI* is not AP in general, it is less vulnerable to manipulation compared to the competitive allocation in exchange economies. We adapt the story in Example 3.6 to a fair division problem.

Example 4.2: An individual is planning to give away a total of 11 kgs of beans and 11 kgs of rice as a food aid. There are 11 potential poor individuals in his neighborhood. The donor announces to these people that he will divide the total among those who appear at his door at a certain time, truly assuming that they will trade afterwards. Beans and rice are necessity for these people, and hence substitutes. Agent 1 prefers beans and the other 10 agents prefer rice. Assume wlog that $u_1 = r_1 + 10b_1$, $u_i = 10r_i + b_i$ for $i = 1, \ldots, 11$.

Recall from Section 3 that if all agents appear at the door, in the final *CEEI* outcome, agent 1 gets 11 kgs of beans with a utility level of 110, and all the other receive 1.1 kg of rice each with a utility level of 11. If only $K \subset S = \{2, \ldots, 11\}$ appears, each agent in S receives $\Omega/|S|$, and hence no group of agents is able to compensate the loss of agent 1 due to his absence. If agent 1 and $K \subset S$ appear, it is easy to check that agent 1 receives 11 kgs of beans and the remaining agents get $11/|S|$ kgs of rice. Apparently, no group has a motivation for manipulation in this case too. Here, the *CEEI* outcome is easy to implement for the donor, efficient and immune to manipulation by absence of some agents.

A plausible rule, well-defined on all finite societies, that is AP but not PM is yet to be explored. However, if we have a maximal society and we adopt allocation schemes as a solution concept, AP solutions violating PM exist. The definition of allocation schemes here is parallel to the one in Section 2.1 i.e., given a problem (N, Ω, R), an allocation scheme assigns an allocation $\mathcal{X}(S) \in \mathbb{R}_+^{S\ell}$ for any problem (S, Ω, R^S) with $S \subseteq N$. The allocation scheme $\mathcal{X}$ in Table 4.1 is AP, but not PM.

Example 4.3: Consider the story in Example 4.2 where the parameters and the preferences are as follows: $\ell = 2$, $\Omega = (4,4)$, $|N| = 3$.

$$u_1 = min\{28x + y, 10y + x\}, u_2 = min\{10x + y, 28y + x\}, u_3 = xy$$

$\boldsymbol{S}$	$\boldsymbol{\mathcal{X}_1}(\boldsymbol{S})$	$\boldsymbol{\mathcal{X}_2}(\boldsymbol{S})$	$\boldsymbol{\mathcal{X}_3}(\boldsymbol{S})$
$\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$	(0.5,1.5)	(1.5,0.5)	(2,2)
$\{\mathbf{1}, \mathbf{2}\}$	(1,3)	(3,1)	$*$
$\{\mathbf{1}, \mathbf{3}\}$	(1,3)	$*$	(3,1)
$\{\mathbf{2}, \mathbf{3}\}$	$*$	(3,1)	(1,3)

Table 4.1

It is easy to check that $\mathcal{X}(S)$ is PO at each S. As $u_3(2,2) > u_3(3,1)$, $\mathcal{X}$ is not PM. To see that $\mathcal{X}$ is not manipulable by $S = \{1,2\}$ via absence of 1, note that agent 2 gets $\mathcal{X}_2(\{2,3\}) = (3,1)$. Then, agent 1 can get a maximum of 13 unit of utility in any redistribution of $(3,1)$ among agents 1 and 2, while agent 1 already gets 15.5 unit of utility at problem N. The argument is similar for the absence of agent 2. Agent 3 will form a coalition with neither agent 1 nor agent 2 as in any case $(1,3)$ or $(3,1)$ will be redistributed among the manipulating coalition, and agent 3 gets at most 3 units of utility while he gets 4 units at the problem N. Hence, $\mathcal{X}$ is AP.

4.3 Models with monetary transfers

In models with quasilinear utilities and monetary transfers, for each problem (N, Ω, R) there is an associated stand-alone TU game defined by the characteristic function $v(S) = max\{\sum_{i \in S} u_i(x_i) : x \in X(S, \Omega, R^S)\}$. This game is not directly useful in AP analysis as we discussed in Section 2.1. However, it is useful to find PM solutions. Moulin (1990b) proved that PM is not compatible with PO in general, in the allocation of a perfectly divisible bundle of goods from $\mathbb{R}_{++}^{\ell}$ (for $\ell > 2$). A similar impossibility was established in Beviá (1996) in the context of indivisible goods where an agent is allowed to receive more than one good. However, introducing a substitutability[11] axiom on the individual preferences and the "joint preference", Moulin (1992) in the context of divisible goods, and Beviá (1996) in the context of indivisible goods showed that the associated TU game is concave. Hence, the Shapley solution

[11] In perfectly divisible case, two goods j, and k are substitutes in the utility function u if marginal utility from consuming good k decreases by an increase in the consumption of good j. The idea is similar for the indivisible case.

is PM by Sprumont (1990). Here, the Shapley solution is simply the set of allocations that yield the final utilities equal to the Shapley value of the stand-alone game. Moulin's result is indeed a bit more general, and also implies that in case of allocating finite objects where each agent receives at most one object (a special case is allocating a single object with monetary transfers), Shapley solution is also PM. In that case if $|N| < |\Omega|$, Theorem 4.1 does not immediately imply that the Shapley solution is AP. However, with some little adjustments it does capture this case also.

Open question 2: Is there an AP rule in the general contexts studied in Moulin (1990b) and Beviá (1996)?

For the special case of distributing two objects (where an agent can get both) Doğan (2013b) shows that on the domain of monotone preferences, PM (and hence AP) solutions always exist. In particular, a hybrid Shapley solution is PM.

The case of single object is equivalent to the well-known airport problems (cost sharing) and this problem admits several interesting population monotonic solutions including the Shapley value, nucleolus and the Dutta-Ray solution of the associated game (see Thomson (2007) for a survey).

Stating the corollaries to Theorem 4.1, we turn back to our primary question: to understand the difference between AP and PM. The difference between the two properties is not obvious in general. However, in the simple model of allocating a single indivisible object, we give a compact characterization of AP rules. This characterization makes it easy to read if a rule is AP or not, moreover it makes the difference between the two properties clearly visible.

4.3.1 A single indivisible object

A single object is to be distributed to a set of individuals N. A well-known example is an inheritance problem where an estate is to be assigned to one of the heirs, and monetary transfers are available to compensate the ones who do not get the object. For simplicity we take $M = 0$ (our results still hold for $M \neq 0$ with minor adjustments). The value of the object for each individual i is $a_i \geq 0$, and a problem is a tuple (N, a) with $a = \{a_i\}_{i \in N}$. A solution to the problem (N, a) is a tuple $\{(\delta_i, t_i)\}_{i \in N}$, where $\delta_i = 0$ for those who do not get the object at the solution, and if j gets the object $\delta_j = 1$. Also, t_i represents the monetary transfer with $\sum_{i \in N} t_i = 0$. Agents preferences are quasilinear in money, i.e., $u_i(\delta_i, t_i) = \delta_i a_i + t_i$ for all $i \in N$. Given a problem (N, a) and a subset $S \subseteq N$, (S, a^S) denotes the restricted problem where $a_i^S = a_i$ for all $i \in S$, and $\bar{a}^S = max_{i \in S}\, a_i$.

As we already discussed, individual rationality (*IR*: that each agent ends up with a non-negative utility) is conceptually a requirement for AP, and PO is a consequence of AP. At a solution satisfying both properties, an agent j with $a_j = \bar{a}^N$ gets the object and compensate the others with monetary transfers such that $t_i \geq 0$ for all $i \neq j$ and $u_j = \bar{a}^N - \sum_{i \in N \setminus j} t_i \geq 0$. Therefore, such a solution induces a unique set of numbers $\{u_i\}_{i \in N}$ with $u_i \geq 0$ and $\sum_{i \in N} u_i = \bar{a}^N$. Conversely, any such set of numbers represents a set of PO and IR solutions among which all individuals are indifferent, and a single solution if there is only one i with $a_i = \bar{a}^N$.

An allocation rule φ assigns a solution to each problem (N, a). Given a problem (N, a) and an allocation rule, $\{(\delta_i(S), t_i(S))\}_{S\subseteq N, i\in S}$ denotes the solutions that the allocation rule assigns to problem (S, a^S) for all $S \subseteq N$, $\{u_i(S)\}_{S\subseteq N, i\in S}$ denotes the induced final utilities.

The main difficulty in analyzing absence-proofness is the complexity of reallocation opportunities after the allocation process. In this model however, the most beneficial reallocation is simple. If the agents from the manipulating coalition who stays in the problem do not receive the object, all we need to know is the total transfer they receive. If one of them, say agent j, receives the object in the sub-problem, the best he can do is to pass the object to an agent, say agent k, with the highest valuation among the agents that stay out, and that is in case $a_k > a_j$.

Proposition 4.3: Let φ be an allocation rule that induces $\{(\delta_i(S), t_i(S))\}_{S\subseteq N, i\in S}$, and $\{u_i(S)\}_{S\subseteq N, i\in S}$ at problem (N, a). Then, the following are equivalent;

(i) φ is AP at (N, a).

(ii) φ is PO at (N, a), and also, if $|N| \geq 3$; $\forall\, S \subset N$ we have; $u_i(N) \leq u_i(S)$ if $\delta_i(S) = 0$, and $u_j(N) - u_j(S) \leq \bar{a}^N - \bar{a}^S$ if $\delta_j(S) = 1$.

Proof:

(i) ⇒ (ii): By Proposition 4.1 φ is PO. Suppose for a contradiction for some $j \in S$ with $\delta_j(S) = 0$ we have $u_j(N) > u_j(S)$, or $\delta_j(S) = 1$ and we have $u_j(N) - u_j(S) > \bar{a}^N - \bar{a}^S$. There are four cases depending on whether $\delta_j(N) = 0$, or $\delta_j(N) = 1$. In each case it is easy to check that $N \setminus j$ can manipulate φ by absence of $N \setminus S$.

(ii) ⇒ (i): Take any $K \subset S \subset N$, let $j \in S$ with $\delta_j(S) = 1$ and (ii) hold. We need to show that $K \cup (N \setminus S)$ does not Pareto improve upon its allocation by leaving $N \setminus S$ out of the problem. Consider first the case that $j \notin K$. Then, $K \cup N \setminus S$ has only the transfers but not the object if $N \setminus S$ leaves, and the total money to be redistributed is $\bar{a}^S - \sum_{i\in S\setminus K} u_i(S)$. By (ii), $\bar{a}^S - \sum_{i\in (S\setminus K)\setminus j} u_i(S) - u_j(S) \leq \bar{a}^N - \sum_{i\in (S\setminus K)\setminus j} u_i(N) - u_j(N) = \sum_{i\in (K\cup(N\setminus S))} u_i(N)$.
Now, consider the case $j \in K$. The total utility of K in problem (S, a^S) is $\bar{a}^S - \sum_{i\in S\setminus K} u_i(S)$. Note that by a redistribution of the transfers and the object $K \cup (N \setminus S)$ can increase this total utility by a maximum of $\bar{a}^N - \bar{a}^S$, and this maximum is reached if there is $k \in N \setminus S$ with $a_k = \bar{a}^N$. By (ii), $\bar{a}^N - \bar{a}^S + \bar{a}^S - \sum_{i\in S\setminus K} u_i(S) \leq \bar{a}^N - \sum_{i\in S\setminus K} u_i(N) = \sum_{i\in (K\cup N\setminus S)} u_i(N)$.
□

Proposition 4.3 makes the difference between AP and PM very clear. PM requires that when a group of agents leave, the utility of none of the agents should decrease. AP, however, allows in such a case, the utility of only one individual (the one who gets the object in the sub-problem) to decrease, and the upper-bound in the change of the utility of this agent is $\bar{a}^N - \bar{a}^S$.

The interesting exercise here is to find out solutions that are AP but not PM. The following solution $\mathcal{X}^{SO}(\cdot)$ (*serial oligarchy*) is an allocation scheme rather than an allocation rule. It is well-defined on a fixed maximal society and all the sub-problems.

Definition 4.4: Fix a maximal society N, and define a linear order on N. Given a problem (N, a); for any problem (S, a^S), assign the object to individual j s.t. j is the first in the order with $a_j = \bar{a}^S$. In the reduced profile $a^{S\setminus j}$, pick the individual j' s.t. j' is the first in the order with $a_{j'} = \bar{a}^{S\setminus j}$. Then, distribute $\bar{a}^S$ equally among this two agents, i.e. $\mathcal{X}_j^{SO} = (1, -\bar{a}^S/2)$, $\mathcal{X}_{j'}^{SO} = (0, \bar{a}^S/2)$, and $\mathcal{X}_i^{SO} = (0,0)$ for all $i \in N \setminus \{j, j'\}$.

Note that the fixed order serves in breaking ties. It is relevant only if there are at least two agents with the highest valuation in the profile, or there is a single agent with the highest valuation, and there are at least two individuals with the second highest valuation.

Proposition 4.4: Given a maximal society N, and a linear order on N, the associated serial oligarchy rule $\mathcal{X}^{SO}(\cdot)$ is AP, but not PM.

Proof: Let $|N| = 3$, and $(a_1, a_2, a_3) = (2,6,10)$. Regardless of the fixed order, final utilities for agent 2 induced by $\mathcal{X}^{SO}(\cdot)$ are $u_2(N) = 5$, and $u_2(\{1,2\}) = 3$. Hence, $\mathcal{X}^{SO}(\cdot)$ is not PM.
Now, fix a maximal society N, a linear order on N, a problem (S, a^S), and suppose a group of individuals leave. In the reduced problem, say S', the utility of only the agent j who gets the object in S' may decrease by definition of $\mathcal{X}^{SO}(\cdot)$ regardless of the order. That happens only in case j does not get the object, but receives a transfer in problem S. In that case, the decrease in his utility is $\left(\bar{a}^S - \bar{a}^{S'}\right)/2$. Hence, by Proposition 4.3, $\mathcal{X}^{SO}(\cdot)$ is AP. ◻

A simple but very compelling fairness property is equal treatment of equals (*ETE*). A solution that satisfies ETE does not discriminate the agents with the same valuation, i.e., for any problem (N, a) the final utilities induced by the rule satisfy $u_i(N, a) = u_j(N, a)$ for any $i, j \in N$ with $a_i = a_j$. PO dictates the assignment of the object to and agent with the highest valuation. In case there are several such agents, this assignment is critical to analyze AP. If in addition to IR and PO we impose ETE, all we need to know is the final utilities of the agents to check for AP. Hence, we can now use *utility distributions* as a solution object, which by definition satisfies PO and IR. All the following results hold for single-valued allocation rules that satisfy ETE.

Definition 4.5: A utility distribution $U(\cdot)$ is a mapping from the set of all problems to $\mathbb{R}_+^n$ s.t. for each (N, a), $\sum_{i \in N} U_i(N, a) = \bar{a}^N$.

Proposition 4.5: Let $U(\cdot)$ be a utility distribution that satisfies ETE. Fix a problem (N, a), and order the individuals s.t. $a_1 \le a_2 \le \cdots \le a_n$. Suppose we add agent j to the problem. In case $a_n = a_{n-1}$ or $a_j < a_n$, AP dictates utility of the pre-existing agents not to increase. In case $a_{n-1} < a_n < a_j$, AP dictates utility of only agent n can increase, and the maximum increase in his utility is $a_j - a_n$. Moreover, these are sufficient conditions for $U(\cdot)$ (any allocation rule φ that yields the same final utlilities with $U(\cdot)$) to satisfy AP.

Proof: Necessity of the conditions directly follows from Proposition 4.3 and ETE. To see the sufficiency, take a problem (N, a), a subproblem (S, a^S), and suppose the conditions above hold. In case $\bar{a}^N = \bar{a}^S$, the condition ii. in Proposition 4.3 trivially holds. In case $\bar{a}^N > \bar{a}^S$, add agents in $N \setminus S$ recursively to the problem (S, a^S), starting with some $j \in N$ with $a_j = \bar{a}^N$. It is easy to see that condition ii. holds here too. ◻

One of the main themes in the fair division literature is the compatibility of the monotonicity properties with different fairness criteria. Alkan (1994) showed that *envy-freeness* (EF) is not compatible with PM. Tadenuma and Thomson (1993) replaced PM with wPM and proposed a rule that satisfy both wPM, and EF. It corresponds to equal division of $\bar{a}^N$ at each problem here, i.e. $u_i(N, a) = \bar{a}^N/n$, for all $i \in N$. To see that this rule is not AP, consider the problem (N, a) with $n = 3$, $(a_1, a_2, a_3) = (2,2,6)$. Note that each agent gets 2 at problem (N, a), while at problem (S, a^S) with $S = \{1,2\}$ each agent gets 1.

Remark 4.1: wPM does not imply AP.

Definition 4.6: A solution $\{(\delta_i, t_i)\}_{i \in N}$ to the problem (N, a) is *envy-free* (*EF*) if for all $i, j \in N$, $u_i(\delta_j, t_j) \geq u_j(\delta_i, t_i)$. An allocation rule φ is EF if it assigns an EF solution to all problems (N, a).

Envy-freeness is a pretty strong condition, especially in this model. It is well-known that EF implies PO (see for example Tadenuma and Thomson (1993)). To derive our next result, it suffices to know two simple properties that EF implies. One of them is ETE, and the second is that any EF allocation assigns an equal share of transfers to those who do not get the object. Both properties follow immediately from Definition 4.6.

Proposition 4.6: There is no allocation rule that satisfy both EF and AP.

Proof: Let φ be EF, and consider the problem (N, a) with $n = 4$ and $(a_1, a_2, a_3, a_4) = (2,2,16,16)$. The unique EF allocation induces the utilities $u_i(N) = 4$ for all $i \in N$, and the unique EF allocation induces the utilities $u_i(S) = 1$ at problem (S, a^S) for $S = \{1,2\}$, and for all $i \in S$. As one of the agents does not get the object at problem S, by Proposition 4.3 φ is not AP. ◻

Moulin (1990b) showed that the key property that causes the incompatibility of PM and EF is the free access upper bound (*FAU*). PM implies FAU, which simply says that the final utility of an agent should be less than his valuation, i.e. $U_i(N, a) < a_i$. A stronger property that PM implies is that U is in the stand-alone core (*SAC*): no coalition S in total can get more than what they get in problem (S, a^S), i.e $\sum_{i \in S} U_i(N, a) \leq \bar{a}^S$. Now, order individuals s.t. $a_1 \leq a_2 \leq \cdots \leq a_n$. Then, in particular, SAC implies $\sum_{i=1}^{k} u_i(N, a) \leq a_k$. We now introduce a similar necessary condition for AP.

Proposition 4.7: Fix a utility distribution $U(\cdot)$ that satisfies ETE, and a problem (N, a). Order the individuals s.t. $a_1 \leq a_2 \leq \cdots \leq a_n$. If φ is AP, then $U(\cdot)$ satisfies the following:

$$\sum_{i=1}^{k} U_i(N, a) \leq a_{k+1} \text{ for all } k \leq n-1 \tag{9}$$

Proof: Let everything be given as in the statement of the proposition. As $U_i(N) \geq 0$ for all $i \in N$, we have the desired inequality for $k = n - 1$. Let $k < n - 1$, and suppose for a contradiction that $\sum_{i=1}^{k} U_i(N,a) > a_{k+1}$. Let $S = \{a_1, \dots, a_{k+1}\}$, consider the problem (S, a^S). Note that $\sum_{i \in S} U_i(S, a^S) = a_{k+1}$. By Proposition 4.5, we have $U_i(N,a) \le U_i(S, a^S)$ for all $i \in \{1, \dots, k\}$. Then, $U_{k+1}(S, a^S) < 0$. □

Now, using the condition (9), we will construct a utility distribution that is AP, but not PM (not even wPM). To avoid notational complexity, we will just introduce and explain it on an example first of all to make it easy to read, and also to explain our choice of distribution when there are several agents with the same valuation. We see below that this choice is critical.

Example 4.4: Let $n = 8$, $(a_1, \dots, a_8) = (2,3,5,6,15,15,15,18)$.

Our utility distribution $\widetilde{U}$ is as follows: Start by assigning the agent with lowest valuation, his value, i.e. $\widetilde{U}_1 = 2$. Continue with agent 2. If his valuation plus $\widetilde{U}_1$ does not exceed a_3 (so that (9) is not violated) assign his valuation to agent 2, $\widetilde{U}_2 = 3$. Now, note that $\widetilde{U}_1 + \widetilde{U}_2 + a_3$ exceeds a_4 , so we give agent 3 the maximum share that does not violate (9), $\widetilde{U}_3 = 1$. It is safe to assign his valuation to agent 4 as $\sum_{i=1}^{3} \widetilde{U}_i + a_4 \le a_5$, hence $\widetilde{U}_4 = 6$. Assigning 5, 6 and 7 equal shares without violating AP is critical. Note that $\sum_{i=1}^{4} \widetilde{U}_i = 12$ and applying the argument we used up to now yields 1.5 for each of these agents. If each gets this share, (9) is not violated and this is the maximum each can get (check (9) for $k = 6$). But suppose we do not have agent 8 initially. Then, agents 5, 6 and 7 gets a share of $(15 - 12)/3 = 1$ in the problem $\left(N \setminus \{8\}, a^{N \setminus \{8\}}\right)$. When agent 8 appears and if we give them 1.5 each, share of 3 of these agents increase and this violates AP as only one of them gets the object in the problem without agent 8. Therefore, what we do here is to give first 7 agents a total of 15 instead of 18. Hence, $\widetilde{U}_5 = \widetilde{U}_6 = \widetilde{U}_7 = 1$. We can generalize this idea as follows: Suppose we have $a_1 \le \dots < a_j = \dots = a_{j+m-1} < \cdots$, and our procedure has already assigned $\{\widetilde{U}_i\}_{i \le j-1}$. Then, $\widetilde{U}_j = \cdots \widetilde{U}_{j+m-1} = \left(a_j - \sum_{i=1}^{j-1} \widetilde{U}_i\right)/m$. We give the remains to agent 8, $\widetilde{U}_8 = 3$. Note that had the vector of valuations be $(2,2,3 \dots)$, we would assign the first two agents 1 each and continue.

Proposition 4.8: $\widetilde{U}$ is AP, and is neither wPM nor in the SAC.

Proof: To see that $\widetilde{U}$ is AP, take a problem (N,a), and order the individuals s.t. $a_1 \le a_2 \le \dots \le a_n$. Consider first adding an agent j to the problem with $a_j \le \bar{a}^N$ and suppose $a_i < a_j < a_{i+1}$. This will not change the share of any agent except that of agents i and $i + 1$, while the share of these agents would possibly decrease but not increase. If $a_i = a_j$, this will not affect the share of agents whose valuation is not equal to a_i , and the share of all agents whose valuation is equal to a_i decreases. Now, suppose $a_j > \bar{a}^N$. This would possibly affect the share of agents with valuation $\bar{a}^N$ only and not the others. If there is more than one agent with valuation equal to $\bar{a}^N$ their share do not change either. Let $a_{n-1} < a_n$, and note that n agents already get a total of $\bar{a}^N$ in problem (N,a). After j arrives, applying (9) for $k = n$, the share of agent n can increase by at most $a_j - \bar{a}^N$. Then, Proposition 4.5 implies that $\widetilde{U}$ is AP.

$\tilde{U}$ is clearly not in the SAC. In order to see that $\tilde{U}$ is not wPM, consider the following problem (N, a) with $n = 5$, $(a_1, \dots, a_5) = (2,3,4,5,10)$. We have $\tilde{U}(N, a) = (2,2,1,5,0)$. Now, let $S = \{1,2,4\}$. We have $\tilde{U}_1(S) = 2$, $\tilde{U}_2(S) = 3$, and $\tilde{U}_4(S) = 0$ at the problem (S, a^S). □

Remark 4.2: By Remark 4.1 and Proposition 4.8, AP and wPM are independent properties.

Appendix

Proof of Proposition 2.4: *Necessity*. Let $\mathcal{X}$ be an APAS. As it is a core selection (hence efficient at each S), and (a_1, b_2) is an optimal pair at game N, by Lemma 2.1, we have $\mathcal{X}_1(N) + \mathcal{X}_2(N) = u_2$, $\mathcal{X}_3(N) = 0$, and also $\mathcal{X}_1(N) + \mathcal{X}_3(N) = \mathcal{X}_1(N) \geq u_1$. Also, by Lemma 2.2, $\mathcal{X}_1(N) = \mathcal{X}_1(1,2)$ and $\mathcal{X}_2(N) = \mathcal{X}_2(1,2)$. Suppose $\mathcal{X}_3(1,3) > \mathcal{X}_2(N)$. Then, b_2 and b_3 would be better off by absence of b_2 at game N.

Sufficiency. Here, the only critical agent is b_2. $\{a_1, b_2\}$ would not be better off if b_2 leaves as they would get at most u_1, while they get together u_2 at game N. $\{b_2, b_3\}$ would not be better off if b_2 leaves as $\mathcal{X}_2(N) \geq \mathcal{X}_3(1,3)$. □

Proof of Proposition 2.6: *Necessity*. Let $\mathcal{X}$ be an APAS. To see $u_3 \geq u_1 + u_2$ holds in Case 1, note that by Proposition 2.4 applied to game $\{2,3,4\}$ we have, $\mathcal{X}_2(2,3) \geq u_2$; and applied to game $\{1,2,3\}$ we have, $\mathcal{X}_3(2,3) \geq u_1$. Then by efficiency of $\mathcal{X}$, $\mathcal{X}_2(2,3) + \mathcal{X}_3(2,3) = u_3 \geq u_1 + u_2$. The argument for Case 2.1 (i) is similar. To see $u_3 + u_4 \geq 3u_2$ holds in Case 1, note that by Proposition 2.4 applied to game $\{1,2,4\}$ we have, $\mathcal{X}_4(1,2,4) \geq u_2$, and hence $u_4 - u_2 \geq \mathcal{X}_1(1,2,4) \geq \mathcal{X}_2(2,4)$. By the same argument for game $\{2,3,4\}$ we have, $u_3 - u_2 \geq \mathcal{X}_4(2,4)$. Then, by efficiency, $u_4 - u_2 + u_3 - u_2 \geq u_2 = \mathcal{X}_2(2,4) + \mathcal{X}_4(2,4)$. The argument for Case 2.1 (ii) is similar. To see $u_3 \geq u_1 + u_2$ holds in Case 2.2, note that applying Prop. 2.4 to game $\{1,2,3\}$ we have $\mathcal{X}_3(2,3) \geq u_1$; and applying it to $\{2,3,4\}$ we have, $\mathcal{X}_2(2,3,4) \geq u_2$. Therefore, $u_3 - u_2 \geq \mathcal{X}_4(2,3,4)$ and $\mathcal{X}_3(2,3,4) = 0$. If $u_1 > u_3 - u_2$, then b_3 and b_4 would be better off by absence of b_4 at game $\{2,3,4\}$. Argument is similar for Case 2.2 (ii). □

Sufficiency. We will show that the allocation scheme defined in Table 2.1 is an APAS, and then define allocation schemes for Case 2.1, and Case 2.2 in Table A.1, and Table A.2, respectively. Proof for those is similar to Case 1, so we omit it. Note that for all subgames that do not appear at the tables $v(S) = 0$. The optimal assignments (not necessarily unique) at game N are $((a_1, b_4), (a_2, b_3))$ in Case 1 and 2.2, and $((a_1, b_3), (a_2, b_4))$ in Case 2.1. Check that $\mathcal{X}(S)$ is efficient at every subgame $S \subseteq N$ in all cases. For all 2-person subgames efficiency and that $v(i) = 0$ for all i implies $\mathcal{X}$ is not manipulable. For 3-person subgames we use Proposition 2.4 to show absence-proofness. We will use the Claim 1 below to prove that $\mathcal{X}$ is not manipualble at game N. Now, fix an allocation scheme $\mathcal{X}$ on (N, v), $T \subseteq N$ and define the set of agents in T whose payoff decrease at game N w.r.t game T as $K(\mathcal{X}; T, N) = \{i \in T: \mathcal{X}_i(T) > \mathcal{X}_i(N)\}$. We say $\mathcal{X}$ is *monotone* from T to N if the set $K(\mathcal{X}; T, N)$ is empty.

Claim 1: Let $\mathcal{X}(N)$ be a core allocation. $\mathcal{X}$ is manipulable by absence of $N \setminus T$ at game N if and only if $\mathcal{X}$ is not monotone from T to N and $K(\mathcal{X}; T, N) \cup N \setminus T$ manipulates $\mathcal{X}$ at game N by absence of $N \setminus T$.

Proof of Claim 1: Let $\mathcal{X}(N)$ be a core allocation, and $K(\mathcal{X}; T, N)$ be empty. Then, for any $K' \subseteq T$ we have, $\sum_{i \in K'} \mathcal{X}_i(T) \leq \sum_{i \in K'} \mathcal{X}_i(N)$, also as $\mathcal{X}(N) \in C(N, v)$, we have $v(N \setminus T) \leq \sum_{i \in N \setminus T} \mathcal{X}_i(N)$. Hence, (1) holds for $N \setminus T \subseteq (K' \cup N \setminus T) \subseteq N$. Now, let $K(\mathcal{X}; T, N)$ be nonempty, and $K' \subseteq T$ s.t. $K' \cup N \setminus T$ manipulates $\mathcal{X}$ at game N by absence of $N \setminus T$. Then, we have $\sum_{i \in K'} \big(\mathcal{X}_i(T) - \mathcal{X}_i(N)\big) > \sum_{i \in N \setminus T} \mathcal{X}_i(N) - v(N \setminus T)$. Note that the expression on the left hand side of the inequality is maximized by $K' = K(\mathcal{X}; T, N)$. □

Case 1: To see that $\mathcal{X}$ in Table 2.1 is an APAS at game $\{1,2,3\}$, check that $\mathcal{X}_3(1,2,3) = \mathcal{X}_3(2,3) = u_3 - u_2 \geq u_1$ by condition (i), $\mathcal{X}_1(1,2,3) = 0$, and $\mathcal{X}_2(2,3) = \mathcal{X}_2(1,2,3) = u_2 \geq u_1/2 = \mathcal{X}_1(1,3)$. To see that $\mathcal{X}$ is an APAS at game $\{1,3,4\}$, check that $\mathcal{X}_1(1,3,4) = \mathcal{X}_1(1,4) = u_4 - u_2 \geq u_1$ by condition (i), $\mathcal{X}_3(1,3,4) = 0$, and $\mathcal{X}_4(1,4) = \mathcal{X}_4(1,3,4) = u_2 \geq u_1/2 = \mathcal{X}_3(1,3)$. To see that $\mathcal{X}$ is an APAS at game $\{2,3,4\}$, check that $\mathcal{X}_2(2,3,4) = \mathcal{X}_2(2,3) \geq u_2$, $\mathcal{X}_4(2,3,4) = 0$, and also $\mathcal{X}_3(2,3) = \mathcal{X}_3(2,3,4) = u_3 - u_2 \geq min\{u_3 - u_2, u_2/2\} = \mathcal{X}_4(1,4)$. To see that $\mathcal{X}$ is an APAS at game $\{1,2,4\}$ check that $\mathcal{X}_4(1,2,4) = \mathcal{X}_4(1,4) \geq u_2$, $\mathcal{X}_2(1,2,4) = 0$. Also, in case $u_3 - u_2 \geq u_2/2$ we have, $\mathcal{X}_1(1,4) = \mathcal{X}_1(1,2,4) = u_4 - u_2 \geq u_2/2 = \mathcal{X}_2(2,4)$, and otherwise $\mathcal{X}_1(1,4) = \mathcal{X}_1(1,2,4) = u_4 - u_2 \geq 2u_2 - u_3 = \mathcal{X}_2(2,4)$ by condition (ii).

To see $\mathcal{X}$ is not manipulable at game N, first note that by Lemma 2.1, $\mathcal{X}(N)$ is in the core of game N as $\mathcal{X}_1(N) + \mathcal{X}_3(N) \geq u_3 - u_2 \geq u_1$ by condition (i), $\mathcal{X}_2(N) + \mathcal{X}_4(N) \geq u_2$, and the optimal pairs get the exact surplus they create. Check that $\mathcal{X}$ is monotone from any T to N as by (i) we have $u_4 - u_2 \geq u_3 - u_2 \geq u_1/2$. Then, by Claim 1 we are done. □

S	$\mathcal{X}_1(S)$	$\mathcal{X}_2(S)$	$\mathcal{X}_3(S)$	$\mathcal{X}_4(S)$
$\{1,2,3,4\}$	$\min\{u_2 - u_1, u_1/2\}$	$u_4 - u_3$	$u_1 - \mathcal{X}_1(N)$	u_3
$\{1,2,3\}$	0	$u_2 - u_1$	u_1	$*$
$\{1,2,4\}$	0	$u_4 - u_3$	$*$	u_3
$\{1,3,4\}$	u_1	$*$	0	$u_3 - u_1$
$\{2,3,4\}$	$*$	$u_4 - u_3$	0	u_3
$\{1,3\}$	$\min\{u_2 - u_1, u_1/2\}$	$*$	$u_1 - \mathcal{X}_1(1,3)$	$*$
$\{1,4\}$	u_1	$*$	$*$	$u_3 - u_1$
$\{2,3\}$	$*$	$u_2 - u_1$	u_1	$*$
$\{2,4\}$	$*$	$u_4 - u_3$	$*$	u_3

Table 2.2

$\boldsymbol{S}$	$\boldsymbol{\mathcal{X}_1(S)}$	$\boldsymbol{\mathcal{X}_2(S)}$	$\boldsymbol{\mathcal{X}_3(S)}$	$\boldsymbol{\mathcal{X}_4(S)}$
$\{\mathbf{1,2,3,4}\}$	$u_4 - u_3$	$u_2 - u_1$	u_1	u_3
$\{\mathbf{1,2,3}\}$	0	$u_2 - u_1$	u_1	$*$
$\{\mathbf{1,2,4}\}$	$u_4 - u_3$	0	$*$	u_3
$\{\mathbf{1,3,4}\}$	$u_4 - u_3$	$*$	0	u_3
$\{\mathbf{2,3,4}\}$	$*$	u_2	0	$u_3 - u_2$
$\{\mathbf{1,3}\}$	$\min\{u_2 - u_1, u_1/2\}$	$*$	$\mathrm{u}_1 - \mathcal{X}_1(1,3)$	$*$
$\{\mathbf{1,4}\}$	$u_4 - u_3$	$*$	$*$	u_3
$\{\mathbf{2,3}\}$	$*$	$u_2 - u_1$	u_1	$*$
$\{\mathbf{2,4}\}$	$*$	u_2	$*$	$u_3 - u_2$

Table 2.3

References

Alkan A (1994) Monotonicity and envyfree assignments. Econ Theory 4(4):605-616

Arin J, Iñarra E (1998) A characterization of the nucleolus for convex games. Games Econ Behav 23(1):12-24

Aumann RJ, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theory 36(2):195-213

Beviá C (1996) Population monotonicity in a general model with indivisible goods. Econ Lett 50(1):91-97

Chichilnisky G, Thomson W (1987) The Walrasian mechanism from equal division is not monotonic with respect to variations in the number of agents. J Public Econ 32:119-124

Doğan E (2013a) Strongly stable and responsive cost sharing solutions for minimum cost spanning tree problems. Mimeo, Rice University

Doğan E (2013b) Population monotonicity in fair division of indivisible goods. Mimeo, Rice University

Dutta B, Ray D (1989) A concept of egalitarianism under participation constraints. Econometrica 57(3):615-635

Foley D (1967) Resource allocation and the public sector. Yale Econ Essays 7:45-98

Hokari T (2002) Monotone-path Dutta-Ray solutions on convex games. Soc Choice Welfare 19:825-844

Hurwicz L (1972) On informationally decentralized systems. In: McGuire CB, Radner R (ed) Decision and organization. North-Holland, Amsterdam, pp 297-336

Iñarra E, José MU (1993) The Shapley value and average convex games. Int J Game Theory 22:13-29

Izquierdo JM, Rafels C (2001) Average monotonic cooperative games. Games Econ Behav 36(2):174-192

Maschler M, Potters J, Tijs S (1992) The general nucleolus and the reduced game property. Int J Game Theory 21:85-106

Moulin H (1990a) Cores and large cores of games when population varies. Int J Game Theory 19:219-232

Moulin H (1990b) Fair division under joint ownership: Recent results and open problems. Soc Choice Welfare 7(2):149-170

Moulin H (1992) An application of the Shapley value to fair division with money. Econometrica 60(6):1331-1349

Moulin, H (1995) Cooperative microeconomics: A game-theoretic introduction. Princeton University Press, Princeton.

Moulin H, Bogomolnaia A (2010) Sharing a minimal cost spanning tree: Beyond the Folk solution. Games Econ Behav 69(2):238-248

Norde H, Reijnierse H (2002) A dual description of the class of games with a population monotonic allocation scheme. Games Econ Behav 41(2):322-343

Norde H, Slikker M (2011) The monoclus of a coalitional game. Games Econ Behav 71(2):420-435

Pazner E, Schmeidler D (1978) Egalitarian-equivalent allocations: A new concept of economic equity. Q J Econ 92(4):671-687

Postlewaite, A (1979) Manipulation via endowments. Rev Econ Stud. 46(2):255-262

Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J Appl Math 17(6):1163-1170

Sertel M, Yıldız M (1962) Double-edged population monotonicity of Walrasian equilibrium: A note on the nature of competition. Math Soc Sci 37(3):235-251

Shapley, LS (1962) A value for n-person games. In: Tucker A, Kuhn H (ed) Contributions to the theory of games, vol II. Princeton University Press, Princeton, pp 307-317

Shapley LS, Scarf H (1974) On cores and indivisibility. J Math Econ 1(1):23-37

Shapley LS, Shubik M (1971) The assignment game I: The core, Int J Game Theory 1(1):111-130.

Sprumont Y. (1990) Population monotonic allocation schemes for cooperative games with transferable utility. Games Econ Behav 2(4):378-394

Sprumont Y (2008) Monotonicity and solidarity axioms in economics and game theory. In: Pattanaik PK, Tadenuma K, Xu Y, Yoshihara N (ed) Rational choice and social welfare: Theory and applications. Springer, pp 71-94

Sönmez T. (1993) Population-monotonicity of the nucleolus on a class of public good problems. Mimeo, Boston College

Tadenuma K, Thomson W (1993) The fair allocation of an indivisible good when monetary compensations are possible. Math Soc Sci 25(2):117-132

Thomson W. (1983) The fair division of a fixed supply among a growing population. Math Oper Res 8(3):319-326

Thomson W. (1995) Population monotonic allocation rules. In: Barnett, WA, Moulin H., Salles M, Schofield N (ed) Social Choice, Welfare, and Ethics. Cambridge University Press, pp 79–124

Thomson W. (2007) Cost allocation and airport problems. Mimeo, University of Rochester.

Thomson W. (2013) New variable-population paradoxes for resource allocation. Soc Choice Welfare. doi:10.1007/s00355-013-0731-5