Strongly Stable and Responsive Cost Sharing Solutions for Minimum Cost Spanning Tree Problems[1]

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Abstract

On the cost sharing solutions to a minimum cost sharing problem, we define a strong stability property *absence-proofness* that implies stand alone core stability. We show that the well-known Bird and Dutta Kar solutions fail this property as well as all non-separable solutions while all population monotonic solutions are strongly stable. We also propose a family of strongly stable solutions that are easy to compute and more responsive than the well-known folk solution to the asymmetries in the cost data.

Key Words: *Core, absence-proofness, population monotonicity, strict ranking*

1 Introduction

We consider a minimum cost spanning tree (*mcst*) problem where agents need to be connected to a source, and there is a fixed cost of the links connecting any two agents and any agent to the source. Agents do not care through which links they are connected. Then, to connect all agents to the source, efficiency requires that the links used in the connection form a spanning tree. Such a tree with the minimal cost (a *mcst*) can be constructed and its cost can be calculated by Prim's algorithm (see Section 2 for details).

Many authors proposed several interesting solutions to distribute the efficient cost. Bird solution (*B*) (Bird 1976) and Dutta Kar solution (*DK*) (Dutta and Kar 2004) are among those. These solutions have been criticized as they lack many desired fairness criteria.[3] However, both

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[3] See Bergantiños and Vidal-Puga (2007) for an extensive axiomatic analysis on these solutions.

solutions satisfy the stand alone core stability, i.e., no group of agents has an incentive to secede from cooperation and construct their own connection to the source. Moreover, they are very easy to calculate when there is a unique *mcst*.

Özsoy (2006) introduced a manipulation idea by merging. A group of agents S leave the scene and covertly connect the source through another agent i, hiding their existence from $N \setminus S$ except i. If the minimal connection cost of S to agent i plus i's cost share at solution φ at the reduced problem $N \setminus S$ is less than the total cost share of $S \cup i$ at the solution φ at the original problem, φ is said to be manipulable by covertly merging. She defined the related stability property covert-merge-proofness and showed that no solution is stable in that sense. Here, we define a similar manipulation idea, and the corresponding stability property absence-proofness. Consider the following problem where the set of agents is $N = \{1,2,3\}$ and ω is the source:

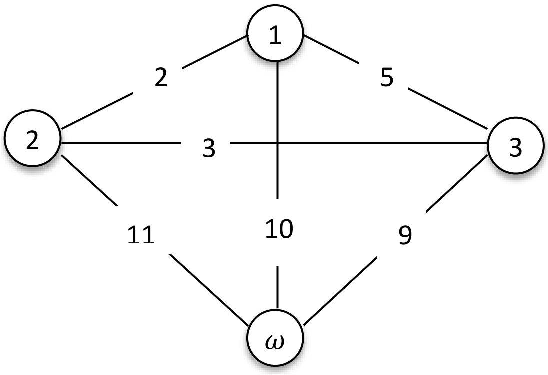

Fig 1

Note that the unique *mcst* is $\{(\omega, 3)(3,2)(2,1)\}$. The Bird solution and the Dutta Kar solution yield the cost allocation vectors $B(N) = (2,3,9)$, and $DK(N) = (9,2,3)$[4]. Suppose now agent 3 leaves or had never appeared in the first place. Then, the solutions yield $B(\{1,2\}) = (10,2,*)$, and $DK(\{1,2\}) = (2,10,*)$ at the reduced problem where only agent 1 and 2 cooperates to connect to the source. Note that the individual connection cost to the source for agent 3 is 9. Consider first the Bird solution. If agents 2 and 3 agree to keep agent 3 away from the scene, they can connect to the source at a total cost of $2 + 9 = 11$, while their total cost share is 12 had 3 not been left aside. A similar argument holds for agents 1 and 3 for the Dutta Kar solution. Hence, both solutions are manipulable by the group of agents $\{2,3\}$ and $\{1,3\}$, respectively, via absence of agent 3.

Doğan (2013) defines the above manipulation argument in various contexts. Here, the general idea is as follows: A solution φ is manipulable by $S \subseteq N$ via absence of $T \subseteq S$ at a problem if the total cost share of S at φ is more than T's own connection cost plus the total cost share of $S \setminus T$ at φ, at the reduced problem $N \setminus T$. If a solution is not manipulable at any problem we call it absence-proof. Absence-proofness (AP) implies core stability by definition; just set $T = S$.

Our first critique is against the solutions that fails separability (SEP). Suppose two groups of agents S and T decide to connect to the source jointly. In case there is no cost saving, i.e. own

[4] See Section 3 for definition of all the properties and the basic solutions that we mention here in the introduction.

connection cost of S plus own connection cost of T is equal to the efficient connection cost of $S \cup T$, cost shares of agents should remain the same. Any solution that violates SEP also violates AP (see Proposition 2).

Our main contribution, by a very simple argument, is that all population monotonic solutions satisfy AP. Population monotonicity (PM) requires that no one should pay more when additional agents arrive. Note that if PM holds, cost share of $S \setminus T$ do not decrease when T leaves. Again by the same argument, T's own connection cost is no less than $\sum_T \varphi_i(N)$.Therefore, PM implies AP.

Several population monotonic, and hence absence-proof, solutions are defined in this context (Feltkamp et al. 1994; Norde et al. 2001; Brânzei et al. 2004; Tijs et al. 2006; Bergantiños and Vidal-Puga 2007; Bergantiños and Lorenzo-Freire 2008; Bogomolania and Moulin 2010; Bergantiños and Vidal-Puga 2012). Among them, the celebrated (as many calls it) *folk solution* singles out, satisfying compelling fairness properties such as *continuity*, *cost monotonicity* and *ranking* (see e.g. Bogomolnaia and Moulin (2010)). It is also known as *equal remaining obligation solution* (Feltkamp et al. 1994) or *P-value* (Brânzei et al. 2004).

Bogomolania and Moulin (2010) criticized the folk solution as in some instances where the cost data strongly suggest that an agent should receive a strictly less cost share from another, it treats the agents equally. Now, write c_{ij} for the cost of connecting agents i, j, and $c_{\omega i}$ for the cost of connecting i to the source ω. Consider the following problem with n agents:

$$c_{\omega i} = c_1 \quad \text{for all } i; \quad c_{ij} = c_2 \quad \text{for all } i, j \geq 2; \quad c_{1i} = 0 \quad \text{for all } i \geq 2 \qquad (1)$$

Bogomolnaia and Moulin (2010) discussed the case $c_1 = 10$, $c_2 = 1$. Note that in the absence of agent 1, the efficient cost of connecting the other agents to the source is 18, and agent 1 comes with 8 units of overall cost saving where the *mcst* is a star and agent 1 is at the central position. Folk solution is reductionist in the sense that it does not take the cost of the links that do not appear in a *mcst* into account, and yields 1 unit of cost share to each agent. Bogomolnaia and Moulin (2010) asks agent 1 to receive a strictly less cost share than to that of others in this case, and formulate the related property *strict ranking 1*. They defined another strict ranking property and a strict cost monotonicity property, and proposed a family of solutions that satisfy all those, together with the basic fairness properties; continuity, cost monotonicity and ranking, except PM. There, whether all these properties are compatible with PM or not is left as an open question.

Norde (2013) is the first to respond this question and proposed the *cost adjusted folk solution* (*CAF*) meeting all the properties mentioned above. *CAF* yields cost allocations very close to the folk solution. In particular, each agent receives a cost share between 97 and 103 percent of his cost share at the folk solution. Moreover, this interval shrinks rapidly as the number of agents n increase. Consider the problem in (1) where $c_1 = c_2 = 100$. Note that agent 1 brings 800 units of cost saving (decreasing the efficient cost from 900 to 100 units). It is natural to ask from a fair allocation to significantly discriminate the cost shares of agent 1 from that of others. One may even argue that agent 1 should receive a negative cost share (a subsidy) (see Trudeau (2012)). However, we follow Bogomolnaia and Moulin (2010), and restrict ourselves to nonnegative cost shares. The *CAF* solution yields approximately a cost share of 9.96 for agent 1 and 10.004 for others while the folk solution yields a 10 unit share to each agent.

In this work, we do not impose any normative principle about how much the solution should favor agent 1 in the above example. Instead, we define a family of solutions which on the one end yields (almost) 10 units to each agent at the above problem and on the other end (almost) 0 to agent 1, depending on a continuous parameter. Moreover, this family carries all the fairness properties Bogomolania and Moulin (2010) asked for, except PM. However, it meets the weaker but more compelling stability property, absence-proofness.

In Section 2 and 3, we give the setting; define basic properties and solutions, respectively. In section 4 we define absence-proofness, give an alternative interpretation of AP, and show that PM implies AP, and AP implies SEP. In Section 5, we define partial solutions on the elementary cost vectors where cost of the links are either 0 or 1, and the extension of these solutions on general cost vectors. We show that for any partial solution satisfying *independence of irrelevant components*, the extended solution is absence proof, and then propose a family of solutions as an alternative to the folk solution.

2 The Setting

Let $N = \{1, \ldots, n\}$ be the set of agents and ω donote the source to which agents need to get connected. There is a nonnegative cost to connect each agent $i \in N$ to the source and to the other agents which is denoted by c_{ik} for $k \in N \cup \{\omega\} \setminus i$. Let $(N \cup \{\omega\})(2)$ denote the set of all non-ordered pairs i, j in set $N \cup \{\omega\}$. Therefore, a *msct* problem is a triple (ω, N, c), where $c = (c_{ij})_{ij \in (N \cup \{\omega\})(2)} \in C$ and C denotes the set of all cost vectors for $N \cup \{\omega\}$. We omit ω as it is fixed and simply speak of the problem (N, c). The reduced problem for a subset $S \subseteq N$ of agents is denoted by (S, c_S), where $c_S = (c_{ij})_{ij \in (S \cup \{\omega\})(2)}$. By abuse of notation we will use (S, c) when we speak of a subporblem.

Given a problem (N, c), an edge $e = e_{ij}$ represents a connection between $i, j \in N \cup \{\omega\}$ with the cost $c_e = c_{ij}$. A spanning tree g is a non-directed graph with n edges that connects all the elements in $N \cup \{\omega\}$ and the cost of g is $c(g) = \sum_{e \in g} c_e$.

The minimal cost of connecting n agents to the source is $v(N, c) = min_{g \in \Gamma(N,c)} c(g)$, where $\Gamma(N, c)$ denotes the set of all spanning trees, and can be computed by Prim's algorithm (Prim 1957): At the first step, among the edges that connect agents to the source, pick an edge $e_{\omega i}$ with the cheapest cost, and say i is connected to the source. At the second step, pick an edge e_{kh} with the minimum cost, where $k \in \{i, \omega\}$ and $h \in N \setminus i$, and say h is connected to the source. At each step, continue to connect a new agent to the agents that are connected in the previous steps or to the source directly using the same method. This procedure ends in n steps and returns a *mcst*. Note that this procedure might not return a unique tree.

A *cost allocation* at problem (N, c) is a vector $y \in \mathbb{R}_+^n$ such that $\sum_{i \in N} y_i = v(N, c)$ and a *solution* φ specifies a cost allocation for each problem. Note that we restrict ourselves to nonnegative cost shares while negative cost shares may be reasonable for some instances (see Trudeau (2012), Kar (2002)).

3 Basic properties and solutions

In this context, there are two interpretations of the most fundamental incentive compatibility property, stand alone core stability. *Strict stand alone cost* of a coalition $S \subseteq N$ ($v(S, c_S)$) is the minimal cost of connecting all agents in S to the source using only the links in $S \cup \{\omega\}$. *Stand alone cost* of S ($\bar{v}(S, c)$) is the minimal cost of connecting agents in S to the source using links in $N \cup \{\omega\}$. The nature of the problem determines the right choice between two cost sharing game v and $\bar{v}$. If we are interested in only core stability, this choice is irrelevant as under the assumption of nonnegative cost shares, both games yield the same set of core allocations (see e.g. Sharkey (1995)). However, this choice is critical here and we will speak of the strict stand alone cost throughout the paper and write $v(S, c)$ by abuse of notation.

Definition 1: Given (N, c), an allocation y is *core stable* if $\sum_{i \in S} y_i \leq v(S, c)$ for all $S \subseteq N$, and a solution φ is a *core selection* (*CS*) if it always assigns a core stable allocation.

Bird (1976) introduced the first core selection in this context. Assume first that there is a unique *mcst* g. To each agent i, *the Bird solution* (*B*) assigns the cost of the edge adjacent to i on the unique path form i to ω in g. If there are multiple *mcst*'s, cost share of i can be calculated by taking the average of the cost shares calculated for each *mcst*.

Cost Monotonicity (*CM*). For all (N, c), (N, c') and $i \in N$, $k \in N \cup \{\omega\}$:

$$\{c_{ik} < c'_{ik} \text{ and } c_e = c'_e \text{ for all } e \neq e_{ik}\} \Rightarrow \varphi_i(N, c) \leq \varphi_i(N, c')$$

Despite the ease of its calculation, Dutta and Kar (2004) criticized the Bird solution as it is not *cost monotonic* (*CM*). Beside its fairness aspect, CM is also considered as an incentive compatibility property. In case it does not hold, agent i would find it profitable to announce the cost of his link more than its actual value if it is private information. If the information is public, violation of CM would kill incentives to decrease the connection costs.

The Dutta Kar solution (*DK*) is calculated through Prim's algorithm. Assume first that there is a unique *mcst* g. Let $\bar{c}_m$ be the cost of the most expensive edge constructed in the first m steps of the algorithm. Suppose agent i is connected to the source at step m, and the edge e_{kh} is constructed at step $m + 1$. Then, *DK* assigns the minimum of c_{kh} and $\bar{c}_m$ to agent i. If there are multiple *mcst*'s, cost share of i can be calculated by taking the average of the cost shares calculated for each *mcst*.

Both solutions are easy to calculate with a unique *mcst*, even for problems with multiple *mcst*'s they satisfy the following axiom.[5]

Polynomial Complexity (*PC*). For all (N, c), $\varphi(N, c)$ is computed by an algorithm polynomial in $n = |N|$.

Naturally, we expect an agent i to pay less than j if i is more efficient in connecting to any other agent or to the source compared to j.

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[5] See e.g. Bogomolnai and Moulin (2010).

Ranking (*RKG*). For all (N,c); $i,j \in N$: $\left\{c_{ik} \leq c_{jk} \text{ for all } k \in N \cup \{\omega\} \setminus \{i,j\}\right\} \Rightarrow \varphi_i \leq \varphi_j$

Note that RKG implies *equal treatment of equals* (ETE) that the cost shares of two agents i, j should be the same whenever $c_{ik} = c_{jk}$ for all $k \in N \cup \{\omega\} \setminus \{i,j\}$. Both *B*, and *DK* obviously satisfy ETE. However, *B* fails RKG while *DK* meets RKG[6]. Hence, we can interpret *DK* as a refinement[7] of *B* not only in terms of CM but also in terms of RKG.

An important critique to these solutions is that in many instances, for a miniscule change in the cost of one link only, there is a substantial change in the cost share of an agent. Also, an agent would be worse off with the arrival of an additional agent. Hence, both solutions fail continuity and population monotonicity. Moreover, they fail separability (weaker than PM): If two sets of agents merge to connect the source jointly and there is no cost saving from the merger, cost shares remain the same for all agents (see e.g. Bergantiños and Vidal-Puga (2007)).

Continuity (*CO*). For all (N,c), $\varphi(N,c)$ is a continuous function of c.

Population Monotonicity (*PM*). For all (N,c), $S \subseteq N$ and $i \in S$, $\varphi_i(N,c) \leq \varphi_i(S,c)$.

Separability (*SEP*). For all (N,c), $S \subseteq N$: $\{v(N,c) = v(S,c) + v(N \setminus S,c)\} \Rightarrow \varphi_i(N,c) = \varphi_i(S,c)$ for all $i \in S$.

The folk solution satisfies all the properties discussed above (see e.g. Bogomolnaia and Moulin (2010)). Among several different descriptions, Bergantiños and Vidal-Puga (2007) uses the *irreducible cost matrix* that is the smallest cost matrix c^* below c such that $v(N,c) = v(N,c^*)$. Then, the *folk solution* is $B(N,c^*)$.

Bogomolnaia and Moulin (2010) criticized the folk solution for it ignores a substantial amount of data in the cost vector and call it reductionist.

Reductionism (*RED*). For all (N,c), $\varphi(N,c) = \varphi(N,c^*)$.

They argue that the folk solution (and any reductionist solution) fails to rank the cost shares strictly when such ranking is compelling in certain cases. They define strict versions of RKG on the domain of cost vectors $\mathcal{D} = \{c \in C : c_{kl} < c_{\omega m} \text{ for all } k,l,m \text{ (not necessarily distinct)}\}$, where connecting any two agent is cheaper than connecting an agent to the source.[8]

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[6] Consider the problem $(c_{\omega 1} c_{\omega 2}, c_{12}) = (2,3,1)$. $B(N,c) = (2,1)$ violates ranking. For *DK* the idea is roughly as follows: Pick i,j s.t. $c_{ik} \leq c_{jk}$ for all $k \in N \cup \{\omega\} \setminus \{i,j\}$. Let m_i, m_j denote the steps of Prim's algorithm at which i and j connect to the source. Let G denote the set of all *mcst*'s, $G^+ \subset G$ be the trees where i's cost share calculated by *DK* method is no more than j's, and $G^- = G \setminus G^+$. It is easy to check that $g \in G^-$ only if $m_j < m_i$. Then, if there is a unique *mcst*, *DK* meets ranking. Suppose there are several and $G^- \neq \emptyset$. Then, we can construct a one to one mapping μ from G^- to G^+ where for each $g \in G^-$, $m_i(\mu(g)) = m_j(g)$ and $m_j(\mu(g)) = m_i(g)$, while the others not necessarily connect at the same step s.t. i's cost share at $\mu(g)$ is no more than j's share at g and j's share at $\mu(g)$ is no less than i's cost share at g. We leave the detailed argument to the curious reader.

[7] Note that when there is a unique *mcst*, both solution yields allocation vectors consisting of same n numbers while they differ in the allocation of these shares to the agents (see the example in Figure 1).

[8] Note that on this domain $v(N,c)$ is equal to the minimum cost of connecting n agents to each other plus $min_{i \in N} c_{\omega i}$. See Bogomolnaia and Moulin (2010) for a detailed justification of these properties on this domain.

Strict Ranking$_1$ (*SRK$_1$*). For any (N, c) s.t. $c \in \mathcal{D}$, and for all $i, j \in N$:

$$\{c_{ik} < c_{jk} \text{ for all } k \in N \setminus \{i, j\} \text{ and } c_{\omega i} \leq c_{\omega j}\} \Rightarrow \varphi_i < \varphi_j$$

Strict Ranking$_2$ (*SRK$_2$*). For any (N, c) s.t. $c \in \mathcal{D}$, and for all $i, j \in N$:

$$\{c_{ik} \leq c_{jk} \text{ for all } k \in N \setminus \{i, j\} \text{ and } c_{\omega i} < c_{\omega j}\} \Rightarrow \varphi_i < \varphi_j$$

Strict Cost Monotonicity (*SCM*). For any (N, c) s.t. $c \in \mathcal{D}$, and for all $i \in N$, $\varphi_i(N, c)$ is strictly increasing in each coordinate c_{ik}, $k \in N \cup \{\omega\} \setminus \{i\}$.

4 Absence-proofness: a strong stability property

Stand alone core stability ensures that no coalition finds it profitable to secede from the cooperation and connect the source on their own. Here, we define an alternative but a related way for a coalition to improve upon their allocation $\varphi(N, c)$, and the associated stability concept. Instead of fully seceding, a coalition $S = K \cup T$ can partially secede and be better off. In particular, T does not appear at the scene and connects to the source using only the links in $T \cup \{\omega\}$. K cooperates with agents in $N \setminus S$ to connect to the source using the links in $(N \setminus T) \cup \{\omega\}$. If the total cost share of S at $\varphi(N, c)$ is strictly more than T's own connection cost plus the total cost share of K at $\varphi(N \setminus T, c)$, S would profit from a partial secession.

Definition 2: A solution φ is *absence-proof* (*AP*)[9], if for all (N, c), $T \subseteq N$, $K \subseteq N \setminus T$:

$$\sum_{i \in K \cup T} \varphi_i(N, c) \leq \sum_{i \in K} \varphi_i(N \setminus T, c) + v(T, c) \tag{2}$$

Remark 1: Note that any AP solution is a core selection; just set $K = \emptyset$.

Generally, stability properties are interpreted as arguments preventing the cooperation from braking up. We now consider the situation from an opposite angle. Suppose two sets of agents S and S' are connecting to the source separately and the links are reconstructed periodically. For example, agents in S live in suburbs a little south of the northwest and S' live in suburbs a little east of the northwest of a big city. The groups S and S' separately carpool to commute to downtown directly. S and S' discovered an alternative option; meet at a point exactly on the northwest and then commute using a single shuttle bus. Suppose this merger yields an overall cost saving, i.e., $\delta(S, S') = v(S, c) + v(S', c) - v(N, c) \geq 0$. Then, AP requires that no coalition from one of these groups has a cost saving more than the total cost saving from the merger. Moreover, that is all AP asks for.

Proposition 1: (*Doğan 2013*) A solution φ is absence-proof if and only if for any (N, c); for all $S, S' \subseteq N$ such that $N = S \cup S'$, $S \cap S' = \emptyset$ and for all $K \subseteq S$ we have,

$$\sum_{i \in K} \big(\varphi_i(S, c) - \varphi_i(N, c)\big) \leq \delta(S, S') \tag{3}$$

Suppose an efficient φ is not a core selection, i.e., $\sum_{i \in S} \varphi_i(N, c) > v(S, c)$ for some S. Then, agents in S unanimously raise a credible objection to the merger between S and $N \setminus S$ as some

[9] We use AP as an abbreviation to both absence-proof and absence-proofness, whichever it fits.

reallocation of $\varphi_i(S,c) = v(S,c)$ to agents in S would make all strictly better off compared to their allocation after the merger. Now, let φ be a core selection but fails (3) for some S, $K \subseteq S$ and $S' = N \setminus S$. Then, agents in $N \setminus K$ unanimously objects to the merger between S and $N \setminus S$, and their objection is credible as now a reallocation of their total cost share before the merger among $N \setminus K$ would make all strictly better off. If (3) holds, there is no credible objection as neither $N \setminus S$ nor $N \setminus K$ unanimously objects.

Following directly from the idea in (3), our first critique is against the solutions that fail SEP, particularly against the Bird and the Dutta Kar solution.

Proposition 2: Any solution that fails SEP also fails AP.

Proof: Fix a core selection φ; (N,c), $S \subseteq N$, $i \in S$ s.t. $v(N,c) = v(S,c) + v(N \setminus S,c)$ and $\varphi_i(N,c) \neq \varphi_i(S,c)$. As φ is CS, $\sum_{i \in T} \varphi_i(N,c) = v(T,c)$ for $T \in \{S, N \setminus S\}$. Then, there is $j \in S$ s.t. $\varphi_j(S,c) - \varphi_j(N,c) > 0 = \delta(S, N \setminus S)$. □

Corollary 1: The Bird and the Dutta Kar solutions fail AP.

Note that for any core selection, when agents T leave the scene and connect on their own, their total will not decrease. So, for $S = K \cup T$ to improve upon their allocation proposed by a core selection at problem (N,c), the cost share of K should decrease in the restricted problem. This fact is summarized in the following proposition.

Proposition 3: Any solution that meets PM also meets AP.

Proof: Let φ satisfy PM. For any $T \subseteq N$, we have $\sum_{i \in T} \varphi_i\,(N,c) \leq \sum_{i \in T} \varphi_i\,(T,c) = v(T,c)$. PM also implies $\sum_{i \in K} \varphi_i\,(N,c) \leq \sum_{i \in K} \varphi_i\,(N \setminus T,c)$. Summing up those two, (2) holds. □

Corollary 2: The folk solution is AP.

Population monotonicity has been interpreted as a normative solidarity concept. As an additional agent always brings nonnegative cost savings, no one should be worse off. By Proposition 3, we can interpret population monotonicity as a strong stability property as well. However, violation of PM does not necessarily mean an opportunity for manipulation. Suppose two groups S, S' decides to connect to the source jointly. Everyone except $i \in S$ has two units of cost saving while i loses 1 unit and i's cost share after the merger is less than $c_{\omega i}$. Then, i cannot convince anyone to object to the merger and he cannot raise a credible objection himself.

In this context, several authors defined families of population monotonic solutions that contain the folk solution. Obligation rules (Tijs et al. 2006) and optimistic weighted Shapley solutions (Bergantiños, Lorenzo-Freire 2008), which is a subset of the obligation rules, are among those. Recently, Bergantiños and Vidal-Puga (2012) introduced a family that contains the obligation rules. This family consists of all population monotonic solutions that satisfy strong cost monotonicity[10] (also known as solidarity). All these solutions are reductionist as solidarity implies RED[11].

[10] For all (N,c), (N,c'): $\{c \leq c'$ and $c_e < c'_e$ for some $e\} \Rightarrow \varphi_i(N,c) \leq \varphi_i(N,c')$ for all $i \in N$.
[11] See e.g. Bogomolnaia and Moulin (2010).

Bogomolnaia and Moulin (2010) pioneered the search for non-reductionist solutions seeking the properties SRK_1, SRK_2, and SCM. Taking CS, CO, RKG and CM as basic fairness standard, they propose families of solutions that satisfy some combinations of $SRK_{1,\ 2}$, SCM and PM. However, none of them satisfy all these properties simultaneously, and they left this issue as an open question. Norde (2013) defined the cost adjusted folk solution which discriminates the cost shares of agents very slightly while a significant discrimination is compelling in some cases. In the next section, we define solutions more responsive to the asymmetries in the cost data compared to *CAF* at the cost of the solidarity aspect of PM.

5 Achieving *SRK₁*, *SRK₂*, *SCM* and *AP*

Here, we follow Bogomolnaia and Moulin (2010) and first define elementary cost vectors $\hat{c}$. Every cost vector c can be written by integrating out these elementary cost vectors in a certain way (see (4)). Then, we define partial solutions that assign allocations to each problem with elementary cost vectors, given (N, c) (see (5)). Finally, we introduce solutions that can be written as an extension of these partial solutions (see (6)). For a more detailed argument on (4), (5), and (6) defined below; and how core stability, polynomial complexity and cost monotonicity extend from partial solutions to their extensions, we refer to Bogomolnaia and Moulin (2010).

A cost vector $\hat{c} \in C^b$ is elementary if $\hat{c}_{ij} \in \{0,1\}$ for all $i, j \in N \cup \{\omega\}$, where C^b represents the set of all elementary cost vectors. For $\hat{c} \in C^b$, $G(\hat{c})$ represents the graph of free edges among the elements in $N \cup \{\omega\}$: $G(\hat{c}) = \{e_{ij}: \hat{c}_{ij} = 0\}$. We say i and j are connected if there is a path between i and j consisting of only free edges. $\mathcal{A}(\hat{c})$ denotes the set of connected components in $G(\hat{c})$ and a particular element is A. Also, $A_i(\hat{c})$ is the connected component i belongs to.

Given any problem (N, c), and $t \in \mathbb{R}_+$ we define $c^t \in C^b$ such that ${c^t}_{ij} = 0$ if $c_{ij} < t$ and ${c^t}_{ij} = 1$ if $c_{ij} \geq t$. Also, let $\bar{c}$ represent the cost of the most expensive edge in $N \cup \{\omega\}$, while $\bar{c}_S$ represent the cost of the most expensive edge in $S \cup \{\omega\}$, i.e. $\bar{c}_S = max_{e \in (S \cup \{\omega\})(2)}\, c_e$. Then, the cost vector c_S can be written as follows:

$$c_S = \int_0^{\bar{c}_S} {c_S}^t dt \tag{4}$$

Let $\psi^b(N, ., c)$ be a *partial solution* that assigns a cost allocation to problem $(N, \hat{c})$ for all $\hat{c} \in C^b$, and for all (N, c):

$$\psi^b(N, \hat{c}, c) \in \mathbb{R}_+^n \quad \text{and} \quad \textstyle\sum_{i \in N} \psi_i^b(N, \hat{c}, c) = v(N, \hat{c}) \tag{5}$$

For any partial solution ψ^b as defined in (5), we can write the extension of this solution to the problem (N, c) as follows:

$$\psi(N, c) = \int_0^{\bar{c}} \psi^b(N, c^t, c) dt \tag{6}$$

Remark 2: Note that for any $r \geq \bar{c}$, as $\sum_{i \in N} \psi_i^b(N, \hat{c}, c) = v(N, \hat{c})$ for all $\hat{c} \in C^b$, we have

$$\psi(N,c) = \int_0^{\bar{c}} \psi^b(N,c^t,c)dt = \int_0^{r} \psi^b(N,c^t,c)dt$$

$\psi(N,c)$ defined by (5) and (6) is a legitimate solution to problem (N,c): $\psi(N,c) \in \mathbb{R}_+^n$ and $\int_0^{\bar{c}} v(N,c^t)dt = v(N,c)$. It is continuous if $\psi^b(N,\hat{c},c)$ is continuous in c for all $\hat{c} \in C^b$, and is of polynomial complexity if $\psi^b(N,\hat{c},.)$ is for all $\hat{c}$. Moreover, ψ is a core selection if (but not only if) for all $\hat{c} \in C^b$, $c \in C$, $A \in \mathcal{A}(\hat{c})$, $i \in A$:

$$\textstyle\sum_{i\in A} \psi_i^b(N,\hat{c},c) = 1 \quad \text{if } \omega \notin A; \qquad \psi_i^b(N,\hat{c},c) = 0 \quad \text{if } \omega \in A \tag{7}$$

Cost monotonicity also extend from ψ^b to ψ in the following way:

Cost Monotonicity. ψ satisfies CM if for all $\hat{c}, \hat{c}' \in C^b$, $c, c' \in C$, $i \in N$, $k \in N \cup \{\omega\} \setminus i$:

$$\{c_{ik} < c'_{ik} \text{ and } c_e = c'_e \text{ for all } e \neq e_{ik}\} \;\Rightarrow\; \psi_i^b(N,\hat{c},c) \leq \psi_i^b(N,\hat{c},c') \text{ and}$$
$$\{\hat{c}_{ik} < \hat{c}'_{ik} \text{ and } \hat{c}_e = \hat{c}'_e \text{ for all } e \neq e_{ik}\} \;\Rightarrow\; \psi_i^b(N,\hat{c},c) \leq \psi_i^b(N,\hat{c}',c)$$

Now, fix two continuous, strictly positive and weakly increasing functions $f, g: \mathbb{R}_+ \longrightarrow \mathbb{R}_{++}$ and consider the following partial solutions $\bar{\psi}^b$ s.t. for any (N,c), $\hat{c} \in C^b$, $A \in \mathcal{A}(\hat{c})$ and $i \in A$:

$$\bar{\psi}_i^b(N,\hat{c},c) = \frac{f(c_{\omega i}) \cdot \prod_{j\in N\setminus i} g(c_{ij})}{\sum_{k\in A} f(c_{\omega k}) \cdot \prod_{j\in N\setminus k} g(c_{kj})} \;\; \text{if } \omega \notin A; \quad \psi_i^b(N,\hat{c},c) = 0 \;\; \text{if } \omega \in A \tag{8}$$

Proposition 4: (*Moulin and Bogomolnaia (2010)*) If f and g are strictly increasing, all solutions $\bar{\psi}$ defined (6) and (8) are core selections meeting CO, CM, RKG, SRK_1, SRK_2, and SCM but fail PM.

Bogomolnaia and Moulin showed if f is strictly increasing and g is constant, $\bar{\psi}$ satisfies all but SRK_1 and SCM. Consider the following 3-person problem: $(c_{\omega 1}, c_{\omega 2}, c_{\omega 3,} c_{12}, c_{13}, c_{23}) = (4,4,2,0,10,20)$. Note that any solution φ meeting SEP and RKG (and hence ETE) yields $\varphi(N,c) = (2,2,2)$.

Remark 3: If g is strictly increasing, all solutions $\bar{\psi}$ defined (6) and (8) fail SEP (and hence AP).

Now we define a property on the partial solutions ψ^b that ensures the extension of the partial solution is absence-proof. Let for any $\hat{c}$, for any $A \in \mathcal{A}(\hat{c})$, $\hat{c}_A = (\hat{c}_{ij})_{ij\in(A\cup\{\omega\})(2)}$.

Definition 3: A partial solution satisfies *independence of irrelevant components* (*IIC*) if for all (N,c), (N',c'); $\hat{c}, \hat{c}' \in C^b$; $A \in \mathcal{A}(\hat{c})$, $A' \in \mathcal{A}(\hat{c}')$ such that $A = A'$, $c_A = c'_A$, and $\hat{c}_A = \hat{c}'_A$:

$$\psi_i^b(N,\hat{c},c) = \psi_i^b(N',\hat{c}',c') \text{ for all } i \in A \setminus \{\omega\}. \tag{9}$$

Lemma 1: Let ψ be the extension of a partial solution ψ^b as defined in (6). If ψ^b satisfies IIC, ψ is a core selection.

Proof: Let ψ^b satisfy IIC. Fix (N,c), $\hat{c} \in C^b$, and $A \in \mathcal{A}(\hat{c})$. For $S = A \setminus \{\omega\}$, we have $\sum_{i\in A} \psi_i^b(N,\hat{c},c) = \sum_{i\in A} \psi_i^b(S,\hat{c}_S,c) = v(S,\hat{c}_S)$. Note that $v(S,\hat{c}_S) = 1$ if $\omega \notin A$, and $v(S,\hat{c}_S) = 0$ if $\omega \in A$. Therefore, (7) holds. □

The idea in IIC is as follows: By Lemma 1, for any $\hat{c} \in C^b$ and the connected component $A(\hat{c})$, ψ^b distributes a total of 1 if $\omega \notin A$ and 0 if $\omega \in A$ to the agents in A. Given a problem (N, c), any partial solution ψ^b that distributes this cost among agents in A only as a function of the cost of links in $A \cup \{\omega\}$ $(c_A, \hat{c}_A)$ satisfy IIC.

Proposition 4: Let ψ be the extension of a partial solution ψ^b. If ψ^b satisfies IIC, ψ is AP.

Proof: Let everything be as in the statement of the proposition. Fix (N, c), $T \subset N$ and let $K_+ = \{i \in N \setminus T: \psi_i(N, c) > \psi_i(N \setminus T, c)\}$. Note that if for some $K \in N \setminus T$, $K \cup T$ is able to manipulate ψ by the absence of T, then $K_+ \cup T$ can manipulate, too. Reordering (2), and by Remark 2, as $\bar{c} \geq \bar{c}_T, \bar{c}_{N\setminus T}$, it suffices to show:

$$\sum_{i\in T} \int_0^{\bar{c}} \left[\psi_i^b(T, {c_T}^t, c) - \psi_i^b(N, c^t, c)\right] dt \geq \sum_{i\in K_+} \int_0^{\bar{c}} \left[\psi_i^b(N, c^t, c) - \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right)\right] dt$$

Define $K_+(t) = \left\{i \in N \setminus T: \psi_i^b(N, c^t, c) > \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right)\right\}$. Note that;

$$\sum_{i\in K_+(t)} \int_0^{\bar{c}} \left[\psi_i^b(N, c^t, c) - \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right)\right] dt \geq$$

$$\sum_{i\in K_+} \int_0^{\bar{c}} \left[\psi_i^b(N, c^t, c) - \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right)\right] dt$$

Now, combining the two inequalities above, it suffices to show that for any $t \leq \bar{c}$, (10) holds.

$$\sum_{i\in T} \psi_i^b(T, {c_T}^t, c) + \sum_{i\in K_+(t)} \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right) \geq \sum_{i\in T\cup K_+(t)} \psi_i^b(N, c^t, c) \tag{10}$$

Let $\mathcal{A}(t) = \mathcal{A}(c^t)$, and $\mathcal{A}^T(t) = \{A \in \mathcal{A}(t): A \cap T \neq \emptyset\}$. Note that by construction of the graph $G(\cdot)$, for any t and $A \in \mathcal{A}(t) \setminus \mathcal{A}^T(t)$ we have $A \in \mathcal{A}\left({c_{N\setminus T}}^t\right)$. Then, by IIC of ψ^b, for any t, $A \in \mathcal{A}(t) \setminus \mathcal{A}^T(t)$ and for all $i \in A \setminus \{\omega\}$ we have $\psi_i^b(N, c^t, c) = \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right)$, and hence, $A \cap K_+(t) = \emptyset$. Then, we can rewrite the inequality (10) as follows:

$$\sum_{i\in \cup_{A\in\mathcal{A}^T(t)}(T\cap A)} \psi_i^b(T, {c_T}^t, c) + \sum_{i\in \cup_{A\in\mathcal{A}^T(t)}(K_+(t)\cap A)} \psi_i^b\left(N \setminus T, {c_{N\setminus T}}^t, c\right) \geq$$

$$\sum_{i\in \cup_{A\in\mathcal{A}^T(t)}[(K_+(t)\cup T)\cap A]} \psi_i^b(N, c^t, c)$$

Note that if for each $A \in \mathcal{A}^T(t)$, the inequality above holds, then taking the union of A in $\mathcal{A}^T(t)$ we have the desired result. Now, fix any $A \in \mathcal{A}^T(t)$.

Case 1: $\omega \notin A$. Agents in $T \cap A$ constitute at least 1 connected component in $G({c_T}^t)$. Then, by Lemma 1, we have:

$$\sum_{i\in(T\cap A)} \psi_i^b(T, {c_T}^t, c) \geq 1 \geq \sum_{i\in[(K_+(t)\cup T)\cap A]} \psi_i^b(N, c^t, c)$$

Case 2: $\omega \in A$. Then, again by Lemma 1, we have:

$$\sum_{i \in [(K_+(t) \cup T) \cap A]} \psi_i^b(N, c^t, c) = 0$$

□

We are inspired by Bogomolnaia and Moulin (2010) for the solutions defined below by (11). Let $f: \mathbb{R}_+ \longrightarrow \mathbb{R}_{++}$ be a continuous, strictly positive[12] and weakly increasing function. Fix a number λ, $1 \leq \lambda \leq \infty$. For all $\hat{c} \in C^b$, and $i \in N$ define $\delta_i(\hat{c}) = \left|\{e_{ij} \in A_i(\hat{c}): \hat{c}_{ij} = 1\}\right|$ that is the number of non-null links i has in A_i, and for all $A \in \mathcal{A}(\hat{c})$, $A^\delta(\hat{c}) = arg\ max_{k \in A}\ \delta_k(\hat{c})$ that is the set of agents in A that has highest number of non-null links. Now, define $\psi_i^{b\lambda}$ successively for $1 \leq \lambda < \infty$ and $\lambda = \infty$ as follows:

$$\begin{aligned} &\psi_i^{b\lambda}(N, \hat{c}, c) = \frac{f(c_{\omega i}) \cdot \lambda^{\delta_i(\hat{c})}}{\sum_{k \in A_i(\hat{c})} f(c_{\omega k}) \cdot \lambda^{\delta_k(\hat{c})}} \text{ if } \omega \notin A_i(\hat{c}); \qquad \psi_i^{b\lambda}(\cdot) = 0 \text{ otherwise} \\ &\psi_i^{b\infty}(N, \hat{c}, c) = \frac{f(c_{\omega i})}{\sum_{k \in A_i^\delta(\hat{c})} f(c_{\omega k})} \text{ if } i \in A_i^\delta(\hat{c}) \text{ and } \omega \notin A_i(\hat{c}); \quad \psi_i^{b\infty}(\cdot) = 0 \text{ otherwise} \end{aligned} \tag{11}$$

Remark 4: If f is constant, ψ^1 is the folk solution.[13]

Proposition 5:

(1) All solutions ψ^λ defined by (6), (11) are core selections meeting CO, RKG, CM, PC and AP.
(2) If $\lambda > 1$, they satisfy SRK_1 but fail PM.
(3) If f is strictly increasing, they all satisfy SRK_2, and they satisfy SCM for $1 < \lambda < \infty$.

Proof: (1) Note that (7) and IIC holds for $\psi_i^{b\lambda}$ for all λ by construction. Also, $\psi_i^{b\lambda}$ is continuous in c for all $\hat{c}$ and it clearly meets PC. Hence, all ψ^λ are core selections meeting CO, PC and AP. For RKG, fix c; $i, j \in N$ such that $c_{ik} \leq c_{jk}$ for all $k \in N \cup \{\omega\} \setminus \{i, j\}$. If $\left|A_j(c^t)\right| \neq 1$, $i \in A_j(c^t)$, and $\delta_i(c^t) \leq \delta_j(c^t)$ for all such t. Also, if $\left|A_j(c^t)\right| = 1$, $\psi_i^{b\lambda}(c^t) \leq \psi_j^{b\lambda}(c^t) = 1$ Then, j receives no less than i at each t in $\psi_i^{b\lambda}(c^t)$ as f is weakly increasing and $\lambda \geq 1$.

For CM, we give the proof for only $\lambda < \infty$. For $\lambda = \infty$, the argument is the same, or simpler. In case we fix $\hat{c}$, we only need to check the case $c_{\omega i}$ is increased to $c'_{\omega i}$ as $\delta_k(\hat{c})$ is fixed for all $k \in N$. Since f is weakly increasing, we are done. Now, suppose the cost of only one link $\hat{c}_e$ is increased from 0 to $\hat{c}'_e = 1$ while the rest of $\hat{c}$, c is fixed. Then, if for one vertex i in e, $\hat{c}_{\omega i} = 0$, $\psi_i^{b\lambda}$ cannot decrease as $\psi_i^{b\lambda}(N, \hat{c}, c) = 0$. Suppose now $\hat{c}_{\omega i} = \hat{c}_{\omega j} = 1$, $\hat{c}_{ij}$ is increased from 0 to $\hat{c}'_{ij} = 1$. Consider first the case $j \in A_i(\hat{c}')$. Let $\gamma_i(\hat{c}) = f(c_{\omega i}) \cdot \lambda^{\delta_i(\hat{c})}$. Note that $\gamma_h(\hat{c}') = \lambda\gamma_h(\hat{c})$ for $h \in \{i, j\}$ and $\gamma_k(\hat{c}') = \gamma_k(\hat{c})$ otherwise. Check that for any nonnegative n numbers $x_1, x_2, \dots x_n$ and $p \geq 1$,

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[12] $\{f(0) = 0$ and f is strictly positive otherwise$\}$ is fine and does not alter our results.

[13] The folk solution is the extension of the following partial solution: $\psi_i^b(N, \hat{c}, \cdot) = \frac{1}{|A|}$ if $\omega \notin A$ and $\psi_i^b(N, \hat{c}, c) = 0$ if $\omega \in A$. See Bogomolnaia and Moulin (2010).

$$[x_1/\sum_{s=1}^{n} x_s] \leq [px_1/(px_1 + px_2 + \sum_{s=3}^{n} x_i)] \tag{12}$$

Now, consider the case $j \notin A_i(\hat{c}') = M$. Then, $A_j(\hat{c}') = M'$ and M partitions $A_i(\hat{c})$ with $|M| = m$ and $|M'| = m'$. Note that $\gamma_i(\hat{c}') = \gamma_i(\hat{c})/\lambda^{(m'-1)}$ and $\gamma_k(\hat{c}') = \gamma_k(\hat{c})/\lambda^{m'}$ for all $i \neq k \in M$. Similarly, $\gamma_j(\hat{c}') = \gamma_j(\hat{c})/\lambda^{m-1}$ and $\gamma_k(\hat{c}') = \gamma_k(\hat{c})/\lambda^{m}$ for all $j \neq k \in M'$. Then,

$$\psi_i^{b\lambda}(N, \hat{c}', \cdot) = \lambda\gamma_i(\hat{c})/[\lambda\gamma_i(\hat{c}) + (\sum_{i \neq k \in M} \gamma_k(\hat{c}))] \geq \gamma_i(\hat{c})/[\sum_{k \in M} \gamma_k(\hat{c})] \geq \psi_i^{b\lambda}(N, \hat{c}, \cdot) \tag{13}$$

(2) For SRK$_1$, let $c \in D$; $i, j \in N$ such that $c_{ik} < c_{jk}$ for all $k \in N \setminus \{i, j\}$ and $c_{\omega i} \leq c_{\omega j}$. We already showed $\psi_i^{b\lambda}(c^t) \leq \psi_j^{b\lambda}(c^t)$ at each t. Consider $k \in \arg max_{l \in N \setminus \{j\}}\, c_{il}$ and recall that for all $t \in (c_{ik}, c_{jk}]$, $\omega \notin A_i(c^t)$. Then, if $t \in (c_{ik}, c_{jk}]$, in case $j \in A_i(c^t)$, we have $\delta_i(c^t) < \delta_j(c^t)$ implying $\psi_i^{b\lambda}(c^t) < \psi_j^{b\lambda}(c^t)$ as f is weakly increasing. In case $j \notin A_i(c^t)$, we have $\psi_i^{b\lambda}(c^t) < \psi_j^{b\lambda}(c^t) = 1$ for $\lambda < \infty$. For $\lambda = \infty$, if $i \in A_i^{\delta}(c^t)$, all $l \in N \setminus \{i, j\}$ are also in $A_i^{\delta}(c^t)$ as $\delta_i(c^t) = 0$.Then, again we have $\psi_i^{b\lambda}(c^t) < \psi_j^{b\lambda}(c^t) = 1$. Therefore, we have the desired result by definition (6).

To see PM fails, let $(N, h; c)$, $h \notin N$ be such that $c_{1h} = c_{\omega i} = 1$ for all $i \in N \cup h$. $c_{1i} = c_{ij} = c_{ih} = 0$ for all $i, j = 2, \ldots, n$. In the absence of h each agent 1 pays $1/n$. When h is present, agent 1 pays $\frac{\lambda}{2\lambda+n-1}$. Therefore, PM requires $\frac{\lambda}{2\lambda+n-1} \leq \frac{1}{n} \Leftrightarrow \lambda \leq \frac{n-1}{n-2}$.

(3) For SRK$_2$, let $c \in D$; $i, j \in N$ such that $c_{ik} \leq c_{jk}$ for all $k \in N \setminus \{i, j\}$ and $c_{\omega i} < c_{\omega j}$. Recall that $\psi_i^{b\lambda}(c^t) \leq \psi_j^{b\lambda}(c^t)$ at each t. Note that $\bar{r} = max_{k,l \in N}\, c_{kl} < r = min_{k \in N}\, c_{\omega k}$ as $c \in D$. Then, for any $t \in (\bar{r}, r\,]$ we have $\delta_k(c^t) = 0$ for all $k \in N$, $\psi_i^{b\lambda}(c^t) = \frac{f(c_{\omega i})}{\sum_{k \in N} f(c_{\omega k})}$, and hence, $\psi_i^{b\lambda}(c^t) < \psi_j^{b\lambda}(c^t)$ if f is strictly increasing. Thus, we have the desired result by definition (6).

For SCM, let f be strictly increasing, $c \in D$. Suppose first $c_{\omega i}$ increase to $\tilde{c}_{\omega i}$ and the rest of c is fixed. Let $\bar{r}, r$ be as defined just above. Note that $c^t = \tilde{c}^t$ for all $t < r$. Then, as f is strictly increasing, $\psi_i^{b\lambda}(c^t) \leq \psi_i^{b\lambda}(\tilde{c}^t)$ for all $t \leq \bar{r}$ and for all λ. Also, for any $t \in (\bar{r}, r\,]$ we have $\delta_k(c^t) = 0$ for all $k \in N$, and hence, $\psi_i^{b\lambda}(c^t) = \frac{f(c_{\omega i})}{\sum_{k \in N} f(c_{\omega k})} < \frac{f(\tilde{c}_{\omega i})}{\sum_{k \in N} f(\tilde{c}_{\omega k})} = \psi_i^{b\lambda}(\tilde{c}^t)$. Then, we have $\psi_i^{\lambda}(c^t) < \psi_i^{\lambda}(\tilde{c}^t)$ by (6). Now, let only c_{ij} increase to $\tilde{c}_{ij} < r$ so that $\tilde{c} \in D$, and the rest of c is fixed. Note that $c^t = \tilde{c}^t$ for all $t \leq \bar{r}$ and $t > \tilde{c}_{ij}$. For all those t as $\delta_k(c^t)$ and $c_{\omega k}$ remains the same for all $k \in N$, $\psi_i^{b\lambda}(c^t) = \psi_i^{b\lambda}(\tilde{c}^t)$. For $t \in (\bar{r}, r\,]$ we have two cases: $j \in A_i(\tilde{c}^t)$ and $j \notin A_i(\tilde{c}^t)$. Then, the proof mimics that of CM and inequalities (12) and (13) hold strictly for $1 < \lambda < \infty$ as $f(c_{\omega k}) > 0$ for all $k \in N$. □

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